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ON THE USE OF QUATERNIONS IN SIMULATION OF RIGID-BODY MOTION

Alfred C. Robinson

Aeronautical Research Laboratory

DECEMBER 1958

WRIGHT AIR DEVELOPMENT CENTER

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DECEMBER 1958

Project No. 7060

WRIGHT AIR DEVELOPMENT CENTER
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FOREWORD

The work covered by this report was done in the System Dynamics Branch, Aeronautical Research Laboratory, under Project 7060, "Flight Dynamics Research and Analysis Facility". Mr. Paul W. Nosker is Project Engineer. This study is part of a continuing program to determine optimum methods of simulation and analysis of the dynamics of air weapon systems. The general subject of quaternions as applied to coordinate conversions has been under investigation for approximately two years, though the bulk of the work reported here was accomplished during the last six months of 1957.

The author wishes to express his appreciation to Mr. Robert T. Harnett and others of the Analog Computation Branch of the Aeronautical Research Laboratory for assistance in the analog simulation portion of the study.

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ABSTRACT

The theory of the four-parameter method is developed with specific application to coordinate conversion in aircraft simulations. This method is compared with the direction cosine method both in a theoretical error analysis and in an example simulation on an analog computer. It is shown that the quaternion method is no more sensitive to multiplier errors than is the direction cosine method, and it requires nearly 30 per cent less computing equipment. In addition, the multiplier bandpass requirement in the four-parameter method is only half as severe as for direction cosines. By every important criterion, the quaternion method is no worse than, and in most cases, better than the direction cosine method.

SECTION I

INTRODUCTION

The problem of motion of a rigid body and the associated one of coordinate conversion are very old ones in the field of classical dynamics. Significant results, dating from the time of Euler (1776) through the introduction and application of matrix methods by Cayley and Klein and others in the last half of the nineteenth century, brought the matter to such a satisfactory state that no significantly new methods or approaches have been found necessary. The development of modern computing machinery makes necessary a re-examination of the various methods from the standpoint of their utility in computational devices. It is not necessarily true that methods which have proven their convenience in the largely analytical manipulations of classical mechanics should prove to be best adapted for numerical or analog computation. Quaternions fell into disuse among physicists about the turn of the present century because matrix and vector methods had proved more useful in the types of investigations then being conducted. The purpose of the present paper is to show that the quaternion approach to coordinate transformation does offer real advantages in the analog simulation of rigid body motion. In recent times Deschamps and Sudduth* have suggested an application for digital computation, and Backus** has proposed them for analog simulation, but in general quaternions are little known among those engaged in simulation of aircraft motions.

The coordinate conversion problem in aircraft and missile simulation is different at least in emphasis from that of classical dynamics. It might be well to state the problem which is of interest and to which the methods explained later will be applied. A missile or aircraft may be considered as a moving coordinate system. Various vectors must be transformed into this coordinate system or out

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*Deschamps, G. A. and W. B. Sudduth, Federal Telecommunications Laboratories, Nutley, New Jersey, Case 26-10707, November 1955.

**Backus, George, Rigid Body Equations - Euler Parameters, Technical Note 6, Advisory Board on Simulation, University of Chicago, November 1951.

Manuscript released by author 15 January 1958 for publication as a WADC Technical Report.

of it into some inertial system. Integrating the equations of motion of the airframe can be made to yield the three components of the coordinate system's angular velocity vector. From the X, Y and Z components (P, Q, R) of this vector in the moving system, it is desired to keep track of the orientation of the coordinate system in such a way that vectors may be transformed in either direction. This means an integration of angular rate to determine angular position.

Fundamental to this procedure is a consideration of how the orientation of the coordinate system is to be specified. During the history of the subject, various methods of doing this have been put forward. All the most useful ones fall into three categories: Euler angles, quaternions, and direction cosines. Of these, the first and last are probably the most familiar to modern readers. In the Euler angle method, the orientation is expressed as the result of three rotations about each of three axes, the rotations being made in a specific sequence. The physical interpretation of a quaternion is a rotation through some angle about some specific fixed axis. The nine direction cosines are simply the cosines of the angles between each of the axes in the moving system with each of the axes of the fixed system. Principal attention here will be given to the quaternion, or four-parameter system. It was first introduced by Euler in 1776, as a result of spherical trigonometry considerations. The elegant quaternion formulation was invented by Hamilton in 1843 as a new kind of algebraic formalism. A matrix formulation was devised by Klein for use in gyroscopic problems and, in this formulation, is usually known as the Cayley-Klein parameters. Each of these three different approaches to the four-parameter system has its own advantages. It has been decided to present at least an outline of all three here. There are two reasons for this: first, there are some propositions which are more easily shown by one development; second, it seems probable that when the reader is offered a choice of method, he will reach an understanding sooner if he can select the method most nearly consonant with his own background.

It will become apparent that this subject presents something of an expositional problem. In order to reach the desired ends, it has been decided to assume that the reader has a knowledge of matrix methods, especially as applied to coordinate conversion in three-dimensional space. As a compromise, a brief introduction to the subject is given in Appendix A, though a more satisfactory treatment is given by Goldstein*. In this report the term "quaternion" has been used to

*Goldstein, Herbert, Classical Mechanics, Addison-Wesley Press, Cambridge, Mass., 1950.

represent the four-parameter method in general. In other cases, it is necessary to use the word to distinguish Hamilton's development from the others. It is hoped that confusion may be kept to a minimum.

There are many different techniques used in present-day aircraft simulations to solve the coordinate conversion problem. The technique is usually adapted to the special requirements of the problem at hand. If most of the rotation takes place about one axis, or if only the gravity vector is to be handled, or if the airframe's rotation is otherwise restricted, valuable simplifications may be effected in the analog equipment required to represent the conversion. It is not the present purpose, however, to investigate all these possibilities. Consideration will be given only to the most general and unrestricted case: that of several complete revolutions about any or all axes. This immediately excludes the Euler angles because of the singular point. The advantages of Euler angles are such, and their popularity is so pervasive, however, as to warrant keeping them in mind. Accordingly, Appendix B gives a brief outline of the Euler Angle system most commonly used in aircraft work, and at appropriate points, comparisons will be made of them with quaternions and direction cosines. In making such comparisons, that form of Euler angle instrumentation whose capabilities most nearly equal those of the quaternion scheme will be assumed. This form has been discussed at some length by Howe* and his figures and results will be used for comparison. In Howe's method, the extent and direction of rotation is unrestricted except for the inevitable singular orientation, and he shows that even this leads to less practical difficulties than one might expect.

It is valuable to keep the Euler angles in mind, but the quaternion method must really stand or fall on its comparison with direction cosines. It has in common with direction cosines the capability of handling completely unrestricted rotations. Accordingly, considerable attention has been devoted to the direction cosine method in this report. Both a theoretical error analysis and a simulation program were done for the cosines in order to provide the most complete possible basis of comparison. They have been done before, but it is difficult to compare results obtained by different investigators on different computing equipment. An attempt was made here to keep the conditions as nearly comparable as possible. Of all the material contained herein, no originality is claimed except for the quaternion error analysis and simulation. Even here, no new techniques were used, with the possible exception of the method of handling multiplier errors. It was felt necessary, however, to include the remaining material in order to introduce and place in context this probably unfamiliar subject.

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*Howe, R. M. and E. G. Gilbert, A New Resolving Method for Analog Computers, WADC Technical Note 55-468, January 1956.

SECTION II

THE EULER PARAMETERS

The earliest formulation of the four-parameter system was given by Euler 1776, though the oldest treatment generally available today is probably that of Whittaker*. It is an essentially geometrical development, but will not be presented as such here. The principal results may be demonstrated with much less labor by use of matrices.

Central to the development of these parameters, and indeed to the four-parameter methods in general, is the proposition known as Euler's theorem, which may be stated as follows: any real rotation may be expressed as a rotation through some angle, about some fixed axis. In other words, regardless of what the rotation history of a body is, once it reaches some orientation, that orientation may be specified in terms of a rotation through some angle (which can be determined) about some fixed axis.

The truth of this proposition is not intuitively obvious, but in any case, it must be shown. Consider a transformation matrix (A). No restrictions are put on (A) other than those which exist for all orthogonal transformation matrices (see Appendix A). Another way of stating Euler's theorem is to say that for every matrix (A) there exists some vector $\overline{R}$ whose components are the same before and after application of (A); in other words, there must be some $\overline{R}$ such that

$$(A)\overline{R} = \overline{R} \tag{1}$$

for any (A). If the components of $\overline{R}$ are designated X, Y and Z, the elements of (A) by a_{mn}, then Equation (1) may be written

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{2}$$

If this matrix equation is expanded in components, a set of linear homogenous

*Whittaker, E. T. Analytical Dynamics, Fourth Edition, Dover Publications, N. Y., 1944.

equations results:

$$(a_{11} - 1)X + a_{12}Y + a_{13}Z = 0,$$

$$a_{21}X + (a_{22} - 1)Y + a_{23}Z = 0, \qquad (3)$$

$$a_{31}X + a_{32}Y + (a_{33} - 1)Z = 0.$$

A necessary and sufficient condition for existence of a non-trivial solution is that the determinant of coefficients be zero. Therefore, it is necessary to show that

$$\begin{vmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{vmatrix} = 0. \qquad (4)$$

This may easily be done making use of the properties of an orthogonal transformation matrix developed in Appendix A. If the above equation is expanded,

$$(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}-a_{31}a_{13}a_{22}-a_{21}a_{12}a_{33}-a_{32}a_{23}a_{11}-1)$$

$$+(a_{11}-a_{22}a_{33}+a_{23}a_{32})+(a_{22}-a_{11}a_{33}+a_{13}a_{31})+(a_{33}-a_{11}a_{22}+a_{21}a_{12})= 0 \qquad (5)$$

The first term vanishes in consequence of the fact that the determinant of the transformation matrix must equal unity (Equation (156)), and the last three terms vanish from the orthogonality conditions of Equation (162). Thus, it is proved that Equation (4) is an identity for any orthogonal (A) and that there exists some vector R which is unchanged by the transformation. This proves Euler's theorem.

Since it has been shown that it is possible to express any rotation as a single rotation about some axis, it is possible to make use of the equivalent rotation to specify orientation. Consider two coordinate systems XYZ and X'Y'Z'. The XYZ system is assumed to be fixed in inertial space, and X'Y'Z' is moving in some arbitrary manner, though both coordinate systems have the same origin. Assume that initially the two systems are coincident. Then the X'Y'Z' system is rotated through an angle μ about an axis which makes angles α, β, γ with the

X, Y, Z axes respectively. It will be noted that this axis of rotation makes the same angles α, β, γ with the X', Y', Z' axes also. It is now necessary to express the transformation matrix in terms of the quantities μ, α, β and γ.

In order to do this, use is made of an additional coordinate system, $X_r Y_r Z_r$, which is fixed in the XYZ system. The X_r axis lies along the axis of rotation, and the Y_r axis is restricted to the XY plane. This would give rise to difficulty if the Z axis is the axis of rotation, but in that case, the Y_r axis could be confined to the XZ plane or the YZ plane, and the final result would be unaltered. At any rate, with the choice indicated, the Y_r axis is always perpendicular to the Z axis. Now the rotation through the angle μ is a rotation through μ about the X_r axis, so the rotation is a very simple one in the $X_r Y_r Z_r$ system. Accordingly, the rotation of the X'Y'Z' system through the angle μ may be viewed as the result of three rotations: (1) rotation of the X'Y'Z' system into coincidence with the $X_r Y_r Z_r$ system; (2) rotation through the angle μ about the X_r axis; (3) the reverse of (1) to restore the original separation of the X'Y'Z' and $X_r Y_r Z_r$ systems. The matrix for each of these transformations will be developed, and then the three may be multiplied together to express the total transformation.

First, the transformation into the $X_r Y_r Z_r$ system will be considered. α, β, and γ are the angles between the new X_r axis and the fixed X, Y and Z axes. Thus, it is seen from Equation (125) that a_{11}, a_{12} and a_{13} are immediately fixed. One other cosine may be established. Recall that the Y_r axis is perpendicular to the Z axis. This means that $a_{23} = 0$. Thus the matrix of the first rotation is partially established,

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} . \qquad (6)$$

Applying the orthogonality conditions, it is possible to deduce that the other elements are

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \mp\cos\beta\csc\gamma & \pm\cos\alpha\csc\gamma & 0 \\ \mp\cos\alpha\cot\gamma & \mp\cos\beta\cot\gamma & \pm\sin\gamma \end{pmatrix} . \qquad (7)$$

The ambiguities in sign may be resolved by making use of the requirement that the matrix above must reduce to the identity matrix when α becomes zero.

The result is

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ -\cos\beta\ \csc\gamma & \cos\alpha\ \csc\gamma & 0 \\ -\cos\alpha\ \cot\gamma & -\cos\beta\ \cot\gamma & \sin\gamma \end{pmatrix}. \tag{8}$$

The second rotation, through the angle μ, about the X_r axis is simply

$$(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{pmatrix}. \tag{9}$$

The last of the three rotations is the inverse of (A) or $(A)^{-1}$. Thus, the general transformation is the result of all three, called (B). It is given by

$$(B) = (A)^{-1}(R)(A). \tag{10}$$

This is a similarity transformation, and, among other things, the spur (sum of the diagonal elements) of a matrix is invariant under a similarity transformation, i.e.,

$$b_{11} + b_{22} + b_{33} = 1 + 2\cos\mu, \tag{11}$$

so the angle of rotation may be obtained directly from the diagonal elements of the transformation matrix. Carrying out the operations of Equation (10) gives

$$(B) = \begin{pmatrix} 1-2\sin^2\frac{\mu}{2}\sin^2\alpha & 2(\sin^2\frac{\mu}{2}\cos\alpha\cos\beta - \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\gamma) & 2(\cos\alpha\cos\gamma\sin^2\frac{\mu}{2} - \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\beta) \\ 2(\sin^2\frac{\mu}{2}\cos\alpha\cos\beta - \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\gamma) & 1-2\sin^2\frac{\mu}{2}\sin^2\beta & 2(\sin^2\frac{\mu}{2}\cos\beta\cos\gamma - \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\alpha) \\ 2(\cos\alpha\cos\gamma\sin^2\frac{\mu}{2} + \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\beta) & 2(\sin^2\frac{\mu}{2}\cos\beta\cos\gamma - \sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\alpha) & 1-2\sin^2\frac{\mu}{2}\sin^2\gamma \end{pmatrix}. \tag{12}$$

If the following substitutions are made,

$$\xi = \cos\alpha\sin\frac{\mu}{2}, \quad \eta = \cos\beta\sin\frac{\mu}{2}, \quad \xi = \cos\gamma\sin\frac{\mu}{2}, \quad \chi = \cos\frac{\mu}{2}, \tag{13}$$

the matrix of (12) becomes

$$B = \begin{pmatrix} \xi^2 - \eta^2 - \zeta^2 + \chi & 2(\xi\eta + \zeta\chi) & 2(\xi\zeta - \eta\chi) \\ 2(\xi\eta - \zeta\chi) & -\xi^2 + \eta^2 - \zeta^2 + \chi^2 & 2(\eta\zeta + \xi\chi) \\ 2(\xi\zeta + \eta\chi) & 2(\eta\zeta - \xi\chi) & -\xi^2 - \eta^2 + \zeta^2 + \chi^2 \end{pmatrix}. \quad (14)$$

These four quantities are called the Euler parameters. It may be seen from their definitions that they obey the relationship

$$\xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1, \quad (15)$$

so they are not all independent. Also, none may lie outside the range ±1.

If the quantities μ, α, β, and γ are known, it is a simple matter to compute the Euler parameters and/or the transformation matrix by the method given above. If, on the other hand, the transformation matrix is given, it is also possible to solve for the four parameters, though difficulties arise. A consideration of these difficulties will shed further light on the nature of the Euler parameters. To begin with, it should be stated that the quantities μ, α, β, and γ cannot be uniquely determined from the transformation matrix. The reason for this is that even though rotation through a certain angle, about a certain axis will produce a definite unambiguous orientation, the reverse is not true. If the orientation is given, there are four separate ways in which it could have been obtained by rotation about a fixed axis. Possibly an example will help to clarify this. Assume that the rotation being considered is a rotation through an angle of + 30° about the + X axis. There are three other ways to get to the same position: (1) a rotation through - 30° about the - X axis; (2) a rotation through - 330° about the + X axis; (3) a rotation through + 330° about the - X axis. A further illustration of the possibilities is given in the table following.

	χ	ξ	η	ζ
Case 1	$+\cos\frac{\mu}{2}$	$+\cos\alpha\sin\frac{\mu}{2}$	$+\cos\beta\sin\frac{\mu}{2}$	$+\cos\gamma\sin\frac{\mu}{2}$
Case 2	$+\cos\frac{\mu}{2}$	$(-\cos\alpha)(-\sin\frac{\mu}{2})$	$(-\cos\beta)(-\sin\frac{\mu}{2})$	$(-\cos\gamma)(-\sin\frac{\mu}{2})$
Case 3	$-\cos\frac{\mu}{2}$	$+\cos\alpha(-\sin\frac{\mu}{2})$	$+\cos\beta(-\sin\frac{\mu}{2})$	$+\cos\gamma(-\sin\frac{\mu}{2})$
Case 4	$-\cos\frac{\mu}{2}$	$(-\cos\alpha)\sin\frac{\mu}{2}$	$(-\cos\beta)\sin\frac{\mu}{2}$	$(-\cos\gamma)\sin\frac{\mu}{2}$

The first two cases lead to the same Euler parameters, and the last two lead to a different set which are the negative of the first. All four sets lead to the same transformation matrix.

The relationship between Euler parameters and direction cosines may be derived by equating terms in Equation (14). The result is

$$
\begin{aligned}
4\chi^2 &= 1 + a_{11} + a_{22} + a_{33}, \\
4\xi^2 &= 1 + a_{11} - a_{22} - a_{33}, \\
4\eta^2 &= 1 - a_{11} + a_{22} - a_{33}, \\
4\zeta^2 &= 1 - a_{11} - a_{22} + a_{33}.
\end{aligned}
\tag{16}
$$

These equations determine the Euler parameters except for sign. The sign must be gotten in another way. From comparison of terms in the matrix it is possible to show that

$$
\begin{aligned}
a_{31} - a_{13} &= 4\chi\eta, \\
a_{12} - a_{21} &= 4\chi\zeta, \\
a_{23} - a_{32} &= 4\chi\xi.
\end{aligned}
\tag{17}
$$

Thus, if χ is assumed to be always positive, the signs of the others may be deduced from Equations (17) unless $\chi = 0$. This is the special case of a 180° rotation. There is an additional ambiguity here because the direction of the axis of rotation and the direction of the rotation are completely unrelated. Either a positive or a negative rotation about either the positive or negative axis will give the same result. For this special case, another means would have to be devised for defining the signs, but it hardly seems worthwhile to go into it here. It is not expected that this will lead to any practical difficulties.

SECTION III

THE CAYLEY-KLEIN PARAMETERS

In this development of the four-parameter system, it is found that a 2x2 complex matrix may be used to represent a real rotation, rather than a 3x3 real matrix. Consider such a matrix (H),

$$(H) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}. \tag{18}$$

The requirement is placed on this matrix that it be unitary, that is to say the product of (H) and its adjoint must yield the unit matrix. The adjoint is the complex conjugate of the transposed matrix. In addition, it is required that the determinant of the matrix (H) have the value +1. The unitary condition allows ±1 for the determinant, so this is an additional requirement. The unitary condition may be written as

$$\begin{pmatrix} h_{11}^* & h_{21}^* \\ h_{12}^* & h_{22}^* \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}$$

Expanding and equating components gives

$$\begin{aligned} h_{11}{}^*h_{11} + h_{21}{}^*h_{21} &= 1, \\ h_{11}{}^*h_{12} + h_{21}{}^*h_{22} &= 0, \\ h_{12}{}^*h_{11} + h_{22}{}^*h_{21} &= 0, \\ h_{12}{}^*h_{12} + h_{22}{}^*h_{22} &= 1. \end{aligned} \tag{20}$$

The second and third equations are the same, being merely complex conjugates of each other. The first and fourth equations have no imaginary component, whereas the second (or third) has both real and imaginary parts. Therefore, the three independent equations contain four conditions. These, together with the determinant requirement that $h_{11}h_{22} - h_{21}h_{12} = +1$ make it possible to determine certain relationships among the four quantities h_{mn}. It may be shown that $h_{22} = h_{11}^*$ and $h_{21} = - h_{12}^*$ so the matrix may be written as

$$(H) = \begin{pmatrix} h_{11} & h_{12} \\ -h_{12}^* & h_{11}^* \end{pmatrix} \qquad (21)$$

The quantities h_{11}, h_{12}, h_{22} are usually referred to as the Cayley-Klein parameters. It will be noted that they are complex numbers. While it is convenient to use them as such in analytical operations (and this is the purpose for which Klein developed them) a physical computer must treat complex numbers in terms of their real and imaginary parts. Therefore, it is convenient to introduce four other quantities defined as follows:

$$\begin{aligned} h_{11} &= e_1 + ie_2, \\ h_{12} &= e_3 + ie_4, \end{aligned} \qquad (22)$$

where the e's are all real numbers, and i is the square root of -1. Using these definitions, the matrix (H) may be written as

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} . \qquad (23)$$

Now consider another complex matrix (P), which has the form

$$(P) = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} , \qquad (24)$$

where x, y and z are real numbers. It will be noted that the matrix (P) is equal to its own adjoint, and thus is said to be self-adjoint or Hermitian. Now consider a transformation of (P) of the form

$$(P)' = (H)(P)(H)^+ \qquad (25)$$

where $(H)^+$ designates the adjoint of (H). Since (H) is unitary, $(H)^+ = (H)^{-1}$, so equation (25) is

$$(P)' = (H)(P)(H)^{-1} . \qquad (26)$$

This is a similarity transformation. It is shown in Appendix A that the determinant of a matrix is invariant under a similarity transformation. It can also be shown that the Hermitian property and the spur are both invariant under a similarity transformation. Therefore, the transformed matrix (P)' must have the form

$$(P)' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix}. \tag{27}$$

The fact that the determinant of (P) must equal the determinant of (P)' gives

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2. \tag{28}$$

If x, y and z are viewed as components of a vector, then Equation (28) is the requirement that the length of the vector remain unchanged. Equation (26) may be written

$$\begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} e_1+ie_2 & e_3+ie_4 \\ -e_3+ie_4 & e_1-ie_2 \end{pmatrix} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \begin{pmatrix} e_1-ie_2 & -e_3-ie_4 \\ e_3-ie_4 & e_1+ie_2 \end{pmatrix}. \tag{29}$$

If the operations of Equation (29) are carried out, it is found that

$$\begin{aligned} x' &= (e_1^2 - e_2^2 - e_3^2 + e_4^2)x - 2(e_1e_2 + e_3e_4)y + 2(e_2e_4 - e_1e_3)z , \\ y' &= 2(e_3e_4 - e_1e_2)x + (e_1^2 - e_2^2 + e_3^2 - e_4^2)y + 2(e_2e_3 + e_4e_1)z , \\ z' &= 2(e_1e_3 + e_2e_4)x + 2(e_2e_3 - e_1e_4)y + (e_1^2 + e_2^2 - e_3^2 - e_4^2)z . \end{aligned} \tag{30}$$

These equations represent a linear transformation between the components of x, y and z, and the components of x' y' and z'. The matrix for this transformation is

$$(A) = \begin{pmatrix} e_1^2-e_2^2-e_3^2+e_4^2 & 2(e_1e_2+e_3e_4) & 2(e_2e_4-e_1e_3) \\ 2(e_3e_4-e_1e_2) & e_1^2-e_2^2+e_3^2-e_4^2 & 2(e_2e_3+e_4e_1) \\ 2(e_1e_3+e_2e_4) & 2(e_2e_3-e_1e_4) & e_1^2+e_2^2-e_3^2-e_4^2 \end{pmatrix}. \tag{31}$$

It may be shown directly that this matrix satisfies the orthogonality conditions, but it is proved also from Equation (28). Equation (31) shows that the nine direction cosines may be expressed in terms of the four e's. If Equations (22) are substituted into Equations (20) it is found that

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1, \tag{32}$$

and therefore, only three of the e's are independent. The identity of these four quantities with the Euler parameters is obvious. Comparison of Equations (31) and (14) gives

$$e_1 = \chi, \quad e_2 = \zeta, \quad e_3 = \eta, \quad e_4 = \xi. \tag{33}$$

An equivalence has been indicated between the real (3x3) matrix (A) and the complex (2x2) matrix (H). It may be shown that this correspondence goes further. Consider the real transformation

$$\bar{r}' = (B)\bar{r}, \tag{34}$$

and let the associated unitary complex matrix be $(H)_1$, so that

$$(P)' = (H)_1(R)(H)_1^+ . \tag{35}$$

Now consider a second transformation (A) with associated $(H)_2$.

$$\bar{r}'' = (A)\bar{r}' ,$$

$$(P)'' = (H)_2(P)'(H)_2^+ , \tag{36}$$

Substituting (34) and (35) into (36) gives

$$\bar{r}'' = (C)\bar{r},$$

$$(P)'' = (H)_2(H)_1(P)(H)_1^+(H)_2^+ . \tag{37}$$

Therefore, if (A)(B) = (C) and $(H)_2(H)_1 = (H)_3$, the above equations become

$$\bar{r}'' = (C)\bar{r},$$

$$(P)'' = (H)_3(P)(H)_3^+ , \tag{38}$$

showing that multiplication of two real 3x3 matrices corresponds to multiplication of the two associated 2x2 complex matrices in the same order. Two types of quantities which correspond in this manner are said to be isomorphic.

It is also possible to view this process of two successive rotations in terms of the e's themselves. Consider one rotation defined by e_1, e_2, e_3 and e_4. After this, another rotation is performed which is described by e_1', e_2', e_3' and e_4'. There is some set of e's called e_1'', e_2'', e_3'', e_4'' which describes the final orientation after the two rotations. This combined set may be found by multiplying the (H) matrices of the two rotations in the correct sequence. The equation is

$$(H)'' = \begin{pmatrix} e_1''+ie_2'' & e_3''+ie_4'' \\ -e_3''+ie_4'' & e_1''-ie_2'' \end{pmatrix} = \begin{pmatrix} e_1'+ie_2' & e_3'+ie_4' \\ -e_3'+ie_4' & e_1'-ie_2' \end{pmatrix} \begin{pmatrix} e_1+ie_2 & e_3+ie_4 \\ -e_3+ie_4 & e_1-ie_2 \end{pmatrix}. \quad (39)$$

Expanding this equation and equating components gives

$$\begin{aligned} e_1'' &= e_1'e_1 - e_2'e_2 - e_3'e_3 - e_4'e_4 , \\ e_2'' &= e_2e_1' + e_2'e_1 + e_3'e_4 - e_4'e_3 , \\ e_3'' &= e_1'e_3 - e_2'e_4 + e_3'e_1 + e_4'e_2 , \\ e_4'' &= e_2'e_3 + e_1'e_4 + e_4'e_1 - e_3'e_2 . \end{aligned} \quad (40)$$

By use of these equations, successive transformations may be handled in terms of the e's directly.

This technique may be used to determine the relationship between the e's and the Euler angles given in Appendix B. The (H) matrix corresponding to each of the Euler angle rotations may be determined, and the three may be multiplied in the correct order to synthesize the complete transformation. Consider first the (H) matrix corresponding to the first Euler angle, given as ψ in Appendix B. From Equation (179) it is seen that the transformation equations are

$$\begin{aligned} x' &= x \cos\psi + y \sin\psi, \\ y' &= -x \sin\psi + y \cos\psi, \\ z' &= z. \end{aligned} \quad (41)$$

Equating coefficients of these equations with like coefficients in Equations (30) gives the nine relations

$$\cos\psi = e_1^2 - e_2^2 - e_3^2 + e_4^2, \quad -\sin\psi = 2(e_3e_4 - e_1e_2), \quad 0 = 2(e_1e_3 + e_2e_4),$$
$$\sin\psi = 2(e_1e_2 + e_3e_4), \quad \cos\psi = e_1^2 - e_2^2 + e_3^3 - e_4^4, \quad 0 = 2(e_2e_3 - e_1e_4), \tag{42}$$
$$0 = 2(e_2e_4 - e_1e_3), \quad 0 = 2(e_2e_3 + e_4e_1), \quad 1 = e_1^2 + e_2^2 - e_3^2 - e_4^2.$$

These equations cannot all be satisfied unless $e_3 = e_4 = 0$. If this is true, then

$$\cos\psi = e_1^2 - e_2^2, \qquad \sin\psi = 2e_1e_2, \qquad e_1^2 - e_2^2 = 1, \tag{43}$$

Solving these equations for e_1 and e_2 gives

$$e_1 = \cos\frac{\psi}{2}, \qquad e_2 = \sin\frac{\psi}{2}, \tag{44}$$

so the (H) matrix corresponding to the ψ rotation is

$$(H)_\psi = \begin{pmatrix} \cos\frac{\psi}{2} + i\sin\frac{\psi}{2} & 0 \\ 0 & \cos\frac{\psi}{2} - i\sin\frac{\psi}{2} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \tag{45}$$

By an exactly similar process, it may be shown that the other two matrices are

$$(H)_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad (H)_\phi = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}. \tag{46}$$

Therefore, the entire transformation, which is the result of all three rotations, is

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} = (H)_\phi \, (H)_\theta \, (H)_\psi . \tag{47}$$

Carrying out the indicated multiplications, and equating components gives

$$\begin{aligned} e_1 &= \cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}, \\ e_2 &= \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}, \\ e_3 &= \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}, \\ e_4 &= \cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}, \end{aligned} \tag{48}$$

SECTION IV

QUATERNIONS

The most brilliant formulation of the four-parameter method was made by Hamilton in 1843. He developed a new type of entity called a "quaternion". It is composed of four parts,

$$q = S + ia + jb + kc, \qquad (49)$$

where S, a, b and c are real numbers, and the indices i, j and k are defined by the following rules;

$$\begin{aligned} i^2 &= -1, & ij &= -ji = k, \\ j^2 &= -1, & jk &= -kj = i, \\ k^2 &= -1, & ki &= -ik = j. \end{aligned} \qquad (50)$$

The conjugate of the quaternion q is

$$q^* = S - ia - jb - kc. \qquad (51)$$

Using the laws for the indices quoted above, it may be easily shown that

$$qq^* = q^*q = S^2 + a^2 + b^2 + c^2, \qquad (52)$$

which is called the length or norm of the quaternion. If this norm is unity, then a special form of quaternion results, a versor. It is possible to make use of these to describe a coordinate transformation. The quantity S is called the real or scalar part of the quaternion, and ia + jb + kc is called the imaginary or vector part. Now assume we have a quaternion whose scalar part is zero. We call this a vector of components X, Y and Z,

$$V = iX + jY + kZ. \qquad (53)$$

Let us examine the operation

$$q^*Vq = V' \qquad (54)$$

where q is a versor. So far there is no particular reason to expect that V' will be a vector, but this turns out to be the case. Equation (54) may be written

$$(S - ia - jb - kc)(iX + jY + kZ)(S + ia + jb + kc) = V'. \qquad (55)$$

When this equation is expanded making use of the rules for indices, the result is

$$\begin{aligned} V' \quad & \text{[illegible]} - c^2] + Y[2cs + 2ab] + Z[2ac - 2sb]\} \\ & \{X[2ab - 2cs] + Y[+S^2 - a^2 - b^2 - c^2] + Z[2as + 2cb]\} \\ & \{X[2\text{[illegible]}] + Y[2bc - 2sa] + Z[s^2 - a^2 - b^2 + c^2]\}. \end{aligned} \qquad (56)$$

This is simply a coordinate transformation whose transformation matrice is

$$\begin{pmatrix} s^2 + a^2 - b^2 - c^2 & 2(cs + ab) & 2(ac - sb) \\ 2(ab - cs) & s^2 - a^2 + b^2 - c^2 & 2(as + cb) \\ 2(ac + sb) & 2(bc - sa) & s^2 - a^2 - b^2 + c^2 \end{pmatrix}. \quad (57)$$

The correlations with matrices derived in the two preceding sections are evidently

$$e_1 = \chi = s, \qquad e_2 = \zeta = c, \qquad e_3 = \eta = b, \qquad e_4 = \xi = a. \quad (58)$$

The matter of two successive rotations may be handled quite easily. Assume that first we transform a vector with the versor q_1.

$$q_1{}^*Vq_1 = V'. \quad (59)$$

Next we apply the versor q_2,

$$V'' = q_2{}^*V'q_2 = q_2{}^*q_1{}^*Vq_1q_2. \quad (60)$$

We now define a new vector $q_1q_2 = q_3$; and wish to find the relationship between q_3 and $q_2{}^*q_1{}^*$. We define $q_4 = q_2{}^*q_1{}^*$. It may be seen that

$$q_2{}^*q_2{}^*q_1{}^* = q_2q_4, \quad (61)$$

and since q_2 is a versor, $q_2q_2{}^* = 1$. Therefore, Equation (61) reduces to

$$q_1{}^* = q_2q_4. \quad (62)$$

Now we apply q_1 on the left,

$$q_1q_1{}^* = q_1q_2q_4 = 1 = q_3q_4, \quad (63)$$

so that q_4 must equal the conjugate of q_3. This means that

$$V'' = q_3{}^*Vq_3. \quad (64)$$

Now observe that the equation $q_3 = q_1q_2$ may be written

$$S_3 - ia_3 - jb_3 - kc_3 = (S_1 + ia_1 + jb_1 + kc_1)(S_2 + ia_2 + jb_2 + kc_2). \quad (65)$$

Expanding this equation and equating components gives

$$\begin{aligned} S_3 &= S_1S_2 - a_1a_2 - b_1b_2 - c_1c_2 , \\ a_3 &= S_1a_2 + S_2a_1 + b_1c_2 - c_1b_2 , \\ b_3 &= S_1b_2 - a_1c_2 + b_1S_2 + c_1a_2 , \\ c_3 &= S_1c_2 + a_1b_2 - b_1a_2 + c_1S_2 . \end{aligned} \tag{66}$$

These equations are identical with Equations (40) which were developed in the same connection by use of the Cayley-Klein parameters. Thus, the quaternion method leads to the same result as the preceeding developments.

SECTION V

INFINITESMAL TRANSFORMATIONS AND RATE OF ROTATION

The preceding sections have dealt with the four-parameter method of specifying the orientation of a coordinate system. As was stated in Section I, however, the primary interest is in determining the orientation from the rate of rotation through a process of integration. Accordingly, it is necessary to relate the rates of change of the four parameters to the rates of rotation of the axis system.

It was shown in Section III that an orthogonal transformation may be represented by a complex matrix having certain properties. It is now of interest to investigate this matrix when an infinitesmal rotation is performed. Let us assume that this infinitesmal rotation consists of a rotation through the angle $\Delta\mu$ about a line which makes angles of α, β and γ with the X, Y and Z axes respectively. Recall that the matrix (H) may be expressed

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix}. \tag{67}$$

Applying the geometrical interpretation of the e's gives

$$(H) = \begin{pmatrix} \cos\frac{\mu}{2} + i\cos\gamma\sin\frac{\mu}{2} & \cos\beta\sin\frac{\mu}{2} + i\cos\alpha\sin\frac{\mu}{2} \\ -\cos\beta\sin\frac{\mu}{2} + i\cos\alpha\sin\frac{\mu}{2} & \cos\frac{\mu}{2} - i\cos\gamma\sin\frac{\mu}{2} \end{pmatrix}. \tag{68}$$

From this, it is possible to see that the infinitesmal rotation may be represented by

$$(H)_\epsilon \begin{pmatrix} 1 + i\frac{\Delta\mu}{2}\cos\gamma & \frac{\Delta\mu}{2}\cos\beta + i\frac{\Delta\mu}{2}\cos\alpha \\ -\frac{\mu}{2}\cos\beta + i\frac{\Delta\mu}{2}\cos\alpha & 1 - i\frac{\Delta\mu}{2}\cos\gamma \end{pmatrix}, \tag{69}$$

since $\cos\frac{\Delta\mu}{2} \sim 1$, $\sin\frac{\Delta\mu}{2} \sim \frac{\Delta\mu}{2}$.

It is expected that any matrix representing an infinitesmal rotation will differ only slightly from the identity matrix. This is true of the above matrix, and

This may be shown more clearly by writing it as follows:

$$(H)_\epsilon \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta\mu}{2} \begin{pmatrix} i \cos\gamma & \cos\beta + i\cos\alpha \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix} = (I) - (\epsilon). \qquad (70)$$

Now assume that this infinitesmal rotation takes place during a small time interval Δt. If (H) is the matrix at the beginning of the interval, and if (H)' is the matrix at the end of the interval, then the time derivative of (H) may be written as

$$\frac{d}{dt}(H) = \lim_{\Delta t \to 0} \frac{(H)' - (H)}{\Delta t} \qquad (71)$$

The final matrix (H)' may also be viewed as the result of two rotations, first (H) and then $(H)_\epsilon$. In other words, $(H)' = (H)_\epsilon\,(H)$. Putting this into the above equation gives

$$\frac{d}{dt}(H) = \lim_{\Delta t \to 0} \frac{(\epsilon)}{\Delta t}(H) \qquad (72)$$

Since (H) is not affected by the time increment, the limiting process refers only to the quantity $\frac{(\epsilon)}{\Delta t}$,

$$\frac{(\epsilon)}{\Delta t} = \frac{1}{2}\frac{\Delta\mu}{\Delta t} \begin{pmatrix} i\cos\gamma & \cos\beta + i\cos\alpha \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix} \qquad (73)$$

In the limit, the quantity $\frac{\Delta\mu}{\Delta t}$ is simply the scalar magnitude of the angular velocity vector. If P, Q and R are the components of this velocity vector along the X, Y and Z axes, then evidently $\frac{d\mu}{dt}\cos\alpha = P$, $\frac{d\mu}{dt}\cos\gamma = R$, $\frac{d\mu}{dt}\cos\beta = Q$, so that

$$\lim_{\Delta t \to 0} \frac{(\epsilon)}{\Delta t} = \frac{1}{2} \begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix} \qquad (74)$$

Therefore, from Equation (72),

$$\frac{d}{dt}(H) = \frac{1}{2} \begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix} (H). \qquad (75)$$

It is also poss[illegible], a straightforward limiting process, that the time [illegible] also a matrix whose elements are the time derivatives of the elements of the original matrix. Therefore,

$$\begin{pmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix} \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix}. \tag{76}$$

Expanding and equating like components gives

$$\begin{aligned} 2\dot{e}_1 &= - e_4P - e_3Q - e_2R, \\ 2\dot{e}_2 &= - e_3P + e_4Q + e_1R, \\ 2\dot{e}_3 &= + e_2P + e_1Q - e_4R, \\ 2\dot{e}_4 &= + e_1P - e_2Q + e_3R. \end{aligned} \tag{77}$$

These are the equations which would be used to compute the four parameters in an actual simulation. Now if we multiply Equation (76) on the right by the adjoint of (H) the result is

$$\begin{pmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{pmatrix} \begin{pmatrix} e_1 - ie_2 & - e_3 - ie_4 \\ e_3 - ie_4 & e_1 + ie_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} iR & Q - iP \\ -Q - iP & - iR \end{pmatrix}. \tag{78}$$

Again expanding and equating components gives

$$\begin{aligned} P &= 2\,(- e_4\dot{e}_1 - e_3\dot{e}_2 + e_2\dot{e}_3 + e_1\dot{e}_4), \\ Q &= 2\,(- e_3\dot{e}_1 + e_4\dot{e}_2 + e_1\dot{e}_3 - e_2\dot{e}_4), \\ R &= 2\,(- e_2\dot{e}_1 + e_1\dot{e}_2 - e_4\dot{e}_3 + e_3\dot{e}_4). \end{aligned} \tag{79}$$

Thus, if the four parameters and their rates of change are known, the angular velocity may be computed.

SECTION VI

THEORETICAL ERROR ANALYSES

In the preceeding sections, the fundamental theory of the quaternion method has been presented. Before proceeding to an application of the method, it is of interest to study theoretically the errors to be expected. Not only will this give a prediction of the results to be obtained in the simulation, but it will give a better understanding of just how the equations must be instrumented in order to achieve the maximum accuracy of which the computer equipment is capable.

As was mentioned earlier, both quaternion and direction cosines will be simulated, so errors for both were analyzed on much the same basis. It is felt that this is an important part of the demonstration, because without a theoretical error compari on, any differences found in the simulation would be subject to the question o computer malfunction. If simulator results and theoretical error analyses agree with each other, the degree of confidence in the comparison will be much higher. Theoretical error analysis is but little used by analog computer operators, especially in non-linear problems such as this. It turns out, however, that both quaternions and direction cosines lend themselves readily to an analysis of errors and the results obtained agree with observations.

A. Direction Cosine Method

The fundamental equations to be used in generating the direction cosine transformation are given in Appendix A. There are, however, many possible variations which will be discussed briefly. Possibly the most straightforward way would be to solve the nine simultaneous equations and thus generate all nine of the direction cosines by integration. As the solution progresses, however, it is inevitable that errors will accumulate. Some of these errors will be of such nature as to cause the orthogonality conditions (Equations 130) not to be satisfied after a time if, indeed, they were satisfied initially. This may be thought of as a departure of the three axes of the moving system from mutual orthogonality and distortion of the unit length of these axes. Some of the errors arising in the solution will not contribute to this, and these may be thought of as angular drift [illegible], the coordinate system will drift as a whole, and in addition, the unit vectors will change their relative orientations and lengths. These last two difficulties may be eliminated, as will be shown presently, but it should be understood that this is of but little value unless some way can be found for making the drift of the system as a whole tolerable. In an aircraft simulation where a coordinate conversion is used, there generally exists some feedback which will eliminate long-term drift in the coordinate system. As will be shown later, the drift can be reduced to where it is much less than those drift-producing elements in the physical system being simulated, such as airframe misalignment, gyro drift and amplifier noise. If the errors due to rotational drift cannot be corrected, there is not much additional penalty in accepting the errors due to non-orthogonality. In any case, it is advisable to determine in advance how much drift can be allowed in the given application, and to design the coordinate conversion to meet the requirements, using the techniques developed in this section.

The possibility of correcting orthogonality errors was first suggested by Corbett*. Possibly a description of the corrections in physical terms will be the most instructive. It may be seen from the material presented in Appendix A, that a physical interpretation may be placed on the rows and columns of the transformation matrix (A). The elements of the first row, for instance, may be considered as the three components of the unit vector $\vec{i'}$ along the three unprimed (fixed) axes. Similarly the elements of the first column may be viewed as the three components of the unit vector $\vec{i}$ along the three primed (moving) axes. Both $\vec{i}$, $\vec{j}$, $\vec{k}$ and $\vec{i'}$, $\vec{j'}$, $\vec{k'}$ coordinate vectors are orthonormal sets. These facts may be written as

$$\begin{array}{llll} \vec{i}\cdot\vec{i} = 1 & \vec{i}\cdot\vec{j} = 0 & \vec{i'}\cdot\vec{i'} = 1 & \vec{i'}\cdot\vec{j'} = 0 \\ \vec{j}\cdot\vec{j} = 1 & \vec{j}\cdot\vec{i} = 0 & \vec{j'}\cdot\vec{j'} = 1 & \vec{j'}\cdot\vec{i'} = 0 \\ \vec{k}\cdot\vec{k} = 1 & \vec{k}\cdot\vec{i} = 0 & \vec{k'}\cdot\vec{k'} = 1 & \vec{k'}\cdot\vec{i'} = 0 \end{array} \qquad (80)$$

These are vector equations, and may, therefore, be expanded in any coordinate system. Expanding the first six in the primed system and the last six in the unprimed system gives . . . .

*Unfortunately, this work has not been generally available. The first published document is WADC Technical Report 57-425 Stabilization of Computer Circuits, November 1957.

$$
\begin{aligned}
a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1, \\
a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1, \\
a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1, \\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0, \\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0, \\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0, \\
a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1, \\
a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1, \\
a_{31}^2 + a_{32}^2 + a_{33}^3 &= 1, \\
a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0, \\
a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= 0, \\
a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0.
\end{aligned}
\qquad (81)
$$

The first six of these will be recognized as the orthogonality conditions given in Appendix A. The last six may be seen to be the same conditions for the matrix $(A)^{-1}$. All twelve equations must be satisfied by any real orthogonal transformation matrix.

The general procedure is to compute the direction cosines by integration, take the computed quantities and perform the operations of some of the Equations (81). If the equations (81) are not satisfied, the error is used to modify the direction cosines until they are. It appears to be necessary to compute at least six of the direction cosines by integration. Several schemes were tried for getting by with less than six, but none of these [illegible]re stable. Assume for instance that the three components of [illegible] [illegible]ed system, and the three components of $\vec{j}$ in the [illegible] [illegible]omputed. The three components of $\vec{k}$ may then be computed without integrations from the relationship that $\vec{k} = \vec{i} \times \vec{j}$. If six direction cosines are computed by integration, then three of Equations (81) must be used to eliminate redundancy. If nine cosines are integrated, then six auxiliary equations must be used. Both possibilities were proposed by Corbett*, and an improved version of the second by Howe**. The first alternative (called the two-vector method) was the one chosen for use here. It was selected because it uses less simulation equipment, though the advantage over Howe's three-vector version is not great. Both require 36 multiplications, though the two-vector method requires only six integrators rather than nine.

The two-vector method may be described as follows. The three components of $\vec{i}$ in the primed system are computed by integration through use of the $\dot{a}_{11}$, $\dot{a}_{21}$ and $\dot{a}_{31}$ differential equations. Then a normalizing circuit is added to keep the length of the vector unity. In addition, the components of $\vec{j}$ in the primed system are computed. These are a_{12}, a_{22} and a_{32}. A correction is added to keep this vector's length equal to unity; another correction is added to keep this vector normal to the first. Finally, the components of $\vec{k}$ are computed from the equation $\vec{k} = \vec{i} \times \vec{j}$. The complete set of equations to be solved is

$$
\begin{aligned}
\dot{a}_{11} &= a_{21}R - a_{31}Q - k_1 a_{11} (1 - \vec{i} \cdot \vec{i}), \\
\dot{a}_{21} &= a_{31}P - a_{11}R - k_1 a_{21} (1 - \vec{i} \cdot \vec{i}), \\
\dot{a}_{31} &= a_{11}Q - a_{21}P - k_1 a_{31} (1 - \vec{i} \cdot \vec{i}), \\
\dot{a}_{12} &= a_{22}R - a_{32}Q - k_1 a_{12} (1 - \vec{j} \cdot \vec{j}) - k_2 a_{11} (\vec{i} \cdot \vec{j}), \\
\dot{a}_{22} &= a_{32}P - a_{12}R - k_1 a_{22} (1 - \vec{j} \cdot \vec{j}) - k_2 a_{21} (\vec{i} \cdot \vec{j}), \\
\dot{a}_{32} &= a_{12}Q - a_{22}P - k_1 a_{32} (1 - \vec{j} \cdot \vec{j}) - k_2 a_{31} (\vec{i} \cdot \vec{j}), \\
a_{13} &= a_{21}a_{32} - a_{22}a_{31}, \\
a_{23} &= a_{12}a_{31} - a_{11}a_{32}, \\
a_{33} &= a_{11}a_{22} - a_{12}a_{21}.
\end{aligned}
\tag{82}
$$

*Corbett, op. cit.

**Howe, R. M. Coordinate Systems and Methods of Coordinate Transformations for Dimensional Flight Equations Proceedings of the First Flight Simulation Symposium, November 56, WSPG Special Report 9, White Sands Proving Ground.

where

$$\vec{i}\cdot\vec{i} = a_{11}^2 + a_{21}^2 + a_{31}^2 \quad \vec{j}\cdot\vec{j} = a_{12}^2 + a_{22}^2 + a_{32}^2,$$

$$\vec{i}\cdot\vec{j} = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}.$$

It will be observed that there are six differential equations, and three algebraic equations, so that six integrators will be required. The k_1 terms are for correction of the length of the vectors and the k_2 terms are to retain orthogonality. The k_1 and k_2 are arbitrary gain factors which will be discussed later. It appears that the above method is a slight improvement on the version given by Corbett in that the orthogonality correction is added to only one vector rather than to both. The entire drift about the z axis, then, is determined by the drift of $\vec{i}$ alone. If the correction were fed into both vectors, the total drift would be higher in cases where the two vectors tend to drift in the same direction.

The above equations are idealizations. The computer which is used to instrument them will be solving approximations of these equations. The differences give rise to errors in the solutions which will now be considered. For convenience, all errors are divided into two categories, static and dynamic. The dynamic errors are associated with the fact that the actual equations the computer is solving are of higher order than the ideal, and the static errors arise from errors in resistance, capacitance, pot settings, and the like. These two types of errors must be treated by different means. The dynamic errors will be considered first.

It is assumed that the only dynamic effect of importance is the bandpass of the multiplier. Amplifiers and integrators should be at least one order of magnitude better than multipliers in this respect, so the assumption appears sound. A simplified analysis will illustrate the important issues. Consider the equations for components of the vector $\vec{i}$, with correction terms deleted.

$$\dot{a}_{11} = a_{21}R - a_{31}Q,$$

$$\dot{a}_{21} = a_{31}P - a_{11}R, \qquad (83)$$

$$\dot{a}_{31} = a_{11}Q - a_{21}P.$$

Now consider a special case, where $P = R = 0$ and Q is a constant. These equations b

$$a_{11} = a_{31}Q,$$

$$a_{31} = a_{11}Q, \tag{84}$$

$$a_{21} = 0.$$

If a_{21} is initially zero, it will remain so, and this equation may be deleted from the set. If Q is constant, the remaining equations are linear. Taking the Laplace transform of these equations, together with the initial conditions $a_{11} = 1$, $a_{31} = 0$, and the result is

$$Sa_{11} = 1 - a_{31}Q,$$

$$Sa_{31} = a_{11}Q \tag{85}$$

If it is assumed that the transfer function of the multipliers is G(S), then these equations would be

$$Sa_{11} = 1 - a_{31}QG(S),$$

$$Sa_{31} = a_{11}QG(S). \tag{86}$$

It is possible to solve these equations for a_{11} and a_{31},

$$a_{11} = \frac{S}{S^2 + Q^2G^2(S)},$$

$$a_{31} = \frac{QG(S)}{S^2 + Q^2G^2(S)}. \tag{87}$$

Before proceeding further, it is necessary to make some assumption concerning G(S). It should be clear that for any reasonable result, this transfer function should be only slightly different from unity. It must equal unity at very low frequencies, (static errors assumed zero). A reasonable assumption is that its linear, i.e., in its power series representation the non-linear terms are negligible in comparison to the linear term. We have assumed as reasonable

$$G(S) = 1 + \tau_1 S. \tag{88}$$

Substituting this into Equations (87) gives

$$a_{11} = \frac{S}{(S + \tau Q^2 + iQ)(S + \tau Q^2 - iQ)} ,$$

$$a_{31} = \frac{\tau Q(S + \frac{1}{\tau})}{(S + \tau Q^2 + iQ)(S + \tau Q^2 - iQ)} . \qquad (89)$$

Taking the inverse transform gives

$$a_{11} = e^{-\tau Q^2 t} \; (\cos Qt - \tau Q \;\sin Qt),$$

$$a_{31} = e^{-\tau Q^2 t} \; (\sin Qt - T Q \cos Qt). \qquad (90)$$

These may be combined to get the length of the vector,

$$\ell = \sqrt{a_{11}^2 + a_{31}^2} = e^{-\tau Q^2 t} \; (1 - \tau Q \;\sin 2\,Qt). \qquad (91)$$

Several conclusions may be drawn from this. If τ is positive (corresponding to a lead in the transfer function) then the length will decrease. If τ is negative (corresponding to lag), then the length will diverge. The term $\tau Q \sin 2Qt$ represents an oscillatory error of peak magnitude τQ. It will be shown later that the amplitude correction term must be kept as small as possible to avoid angular drift. Therefore, it will probably not be possible to get the correction gain high enough to cut down this oscillatory error term. The only way to keep it small will be to keep τQ small. Let us assume, for instance, that a computing accuracy of 0.1 per cent is desired. This means that $\tau Q \leq 10^{-3}$. It may also be seen that τQ is simply the phase lag (in radians) at the frequency Q. For the example given, the phase lag at the frequency of oscillation should be less than one milliradian, or about 0.06 degrees. The correction circuit will be able to take care of the long term exponential increase quite well, though if the bandpass requirement stated above is met, this source of growth of the vector will be negligible compared with those due to static errors which will be considered next.

As a prelude to studying the effect of static errors, it will be useful to consider a general symbolic diagram of the circuit required to solve the equations. This is because the problem of errors is inseparable from that of scaling, so some sort of scaling must be assumed. This diagram is given in Figure 1. It may be seen that there are 36 multiplications, six of which are independent. There are six integrators, six summing amplifiers, and in any practical circuit, there would have to be a considerable number of inverting and isolation amplifiers as well. As the nature and number of these depends on characteristics of the multipliers being used, they are deleted in this figure. A practical circuit for accomplishing this transformation will be considered later. It is convenient to introduce a special notation for errors in these multipliers. The voltage error in the multiplier which multiplies $100a_{11}$ and $100\frac{R}{R_m}$ is designated ϵ_{11R}. It should be emphasized that this is the actual error in volts, not a ratio or a percentage.

Now assume that the multipliers have infinite bandpass, but do have static errors. It may be seen from the circuit of Figure 1 that the equations for components of the vector $\vec{i}$ are actually

$$\dot{a}_{11} = a_{21}R - a_{31}Q + K_1 a_{11}(1 - \vec{i}\cdot\vec{i}) + R_m \frac{E_{21R}}{100} - Q_m \frac{E_{31Q}}{100} + \frac{K_1}{C_1}\frac{E_{11C}}{100},$$

$$\dot{a}_{21} = a_{31}P - a_{11}R + K_1 a_{21}(1 - \vec{i}\cdot\vec{i}) + P_m \frac{E_{31P}}{100} - R_m \frac{E_{11R}}{100} + \frac{K_1}{C_1}\frac{E_{21C}}{100}, \tag{92}$$

$$\dot{a}_{31} = a_{11}Q - a_{21}P + K_1 a_{31}(1 - \vec{i}\cdot\vec{i}) + Q_m \frac{E_{11Q}}{100} - P_m \frac{E_{21P}}{100} + \frac{K_1}{C_1}\frac{E_{31}}{100}.$$

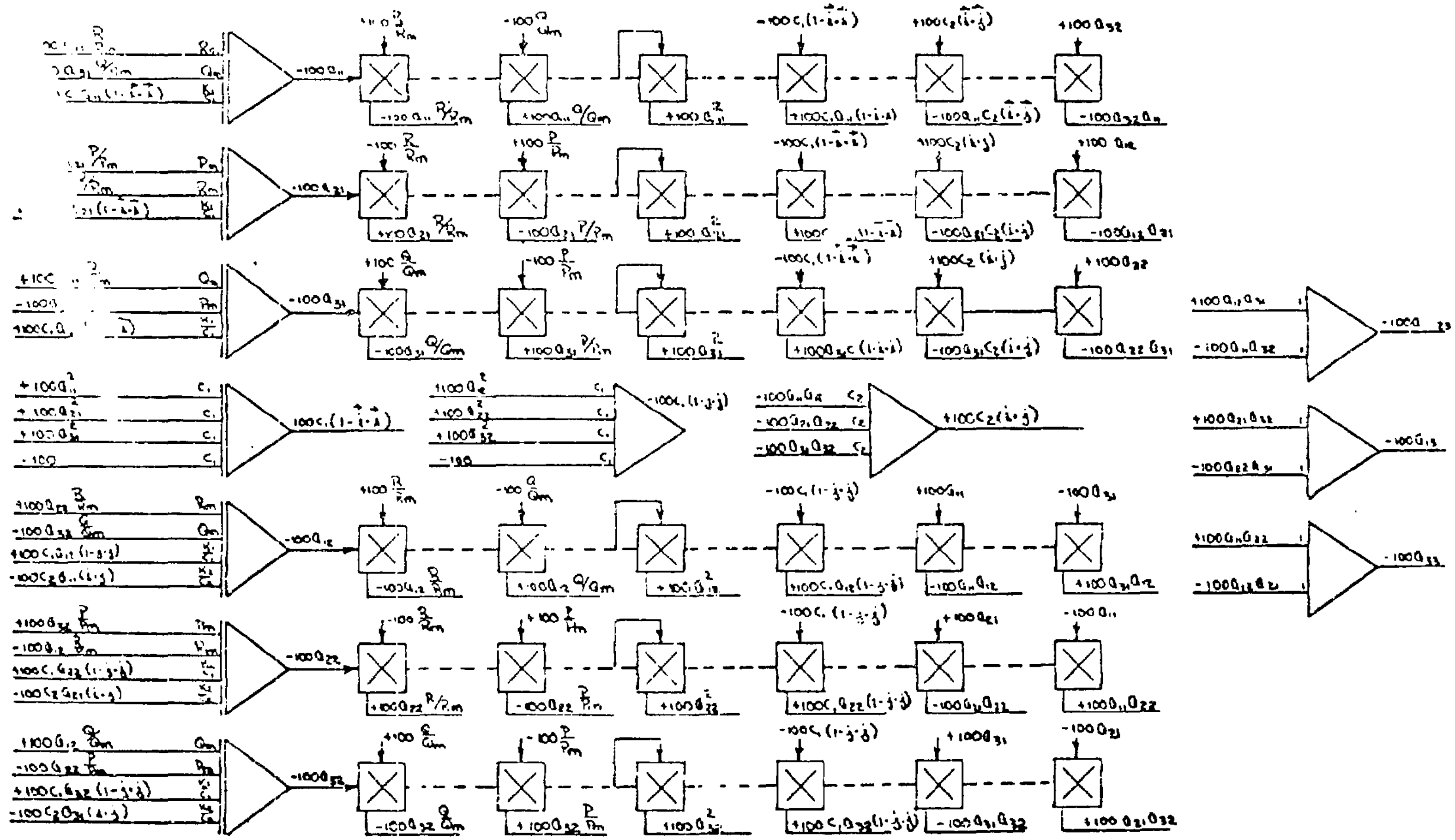
These are the equations which the machine will be solving. It is of interest to investigate certain properties of the solutions to these equations, and to see how they compare with the ideal.

First consider the effect on the length of the unit vector $\vec{i}$. Assume that $K_1 = 0$ so that no corrections are being introduced. The equations become

$$\dot{a}_{11} = a_{21}R - a_{31}Q + R_m \frac{E_{21R}}{100} - Q_m \frac{E_{31Q}}{100},$$

$$\dot{a}_{21} = a_{31}P - a_{11}R + P_m \frac{E_{31P}}{100} - R_m \frac{E_{11R}}{100}, \tag{93}$$

$$\dot{a}_{31} = a_{11}Q - a_{21}P + Q_m \frac{E_{11Q}}{100} - P_m \frac{E_{21P}}{100}.$$



DIRECTION COSINES

Figure 1

The length of $\vec{i}$ is given by $\ell^2 = a_{11}^2 + a_{21}^2 + a_{31}^2$ and, differentiating this expression with respect to time,

$$2\ell\dot{\ell} = 2a_{11}\dot{a}_{11} + 2a_{21}\dot{a}_{21} + 2a_{31}\dot{a}_{31}.$$

The length is close to unity for all cases of interest, so that, to the first order,

$$\dot{\ell} = a_{11}\dot{a}_{11} + a_{21}\dot{a}_{21} + a_{31}\dot{a}_{31}. \tag{94}$$

Substituting Equations (93) into this expression gives

$$\dot{\ell} = a_{11}R_m \frac{E_{21R}}{100} - a_{11}Q_m \frac{E_{31Q}}{100} + a_{21}P_m \frac{E_{31P}}{100} - a_{21}R_m \frac{E_{11R}}{100} \tag{95}$$

$$+ a_{31}Q_m \frac{E_{11Q}}{100} - a_{31}P_m \frac{E_{21P}}{100}.$$

It is now necessary to consider the nature of these errors. In the first place, the error is viewed as a random variable. For a given multiplier, the error which exists is some definite function of the two inputs. This function is more or less repeatable, at least over a short time period, so in this sense it is not a random variable. However, when one considers different multipliers, the error existing at certain values of the inputs is now a function of which multiplier is used and is thus a random variable. These variables are considered independent because there is no reason to expect that the error in one multiplier will influence the error in another.

Two different types of error are considered in the subsequent analysis. In the first type, called "uncorrelated", the error is assumed to be Gaussian and independent of both inputs. In other words, the error existing at any given value of the inputs is assumed to be drawn from a normal distribution of zero mean and variance which is independent of either input and of the particular multiplier being used. This error distribtuion is intended to be consistent with electronic multipliers, though no proper statistical data are available on them. Usually electronic multipliers are adjusted so that the error when either of the inputs are zero is somewhat smaller than otherwise, and there are no data to support using a normal distribution. Until such data are available the above hypotheses are as good as any and more convenient than mos

The second type of error, called "correlated" differs from the first in that the variance of the errors is assumed proportional to one of the inputs and independent of the other. This is intended to represent servomultipliers, and the experimental justification here is somewhat better. Since this is the type of multiplier used in the simulation described later, the error distribution was measured. Results are given in Appendix C, and the normality of the distribution is reasonably well verified. Proportionality of the variance to the voltage across the potentiometer was not checked but it is an inevitable consequence of the nature of servomultipliers.

The drift rate $\dot{\ell}$ of Equation (95) thus becomes a random variable. It is the sum of six independent Gaussian variables, so its variance is the sum of the variances of the individual terms. Thus the standard deviation or square root of the variance is given by

$$\sigma_{\dot{\ell}} = \frac{\sigma_\epsilon}{100}\left[{a_{11}}^2 {R_m}^2 + {a_{21}}^2 {P_m}^2 + {a_{31}}^2 {Q_m}^2 + {a_{21}}^2 {R_m}^2 + {a_{31}}^2 {P_m}^2\right]^{\frac{1}{2}}. \qquad (96)$$

$\sigma_{\dot{\ell}}$ is the standard deviation of $\dot{\ell}$ and σ_ϵ is the standard deviation of multiplier errors. It is not possible to evaluate this expression without knowledge of the particular P_m, Q_m and R_m involved, but it is possible to determine a convenient upper bound. The total maximum angular rate W_m is given by

$$W_m = \sqrt{Q_m^2 + P_m^2 + R_m^2}\,.$$

Making use of this fact together with the normality relation ${a_{11}}^2 + {a_{21}}^2 + {a_{31}}^2 = 1$, it may be seen that

$$\sigma_{\dot{\ell}} \leq \frac{\sigma_\epsilon}{100}\sqrt{2}\; W_m\,. \qquad (97)$$

In the foregoing, errors of the first, or uncorrelated type have been assumed. The drift rate tends to be proportional to the full-scale velocity, regardless of what velocity actually exists.

This is not true with the second or correlated type of error. Observe that each of the multipliers is used to multiply a direction cosine by one of the angular velocity components, P, Q or R. Consider, for example the multiplier which generates the first error term in equation (95); the one multiplying a_{11} by R.

If the direction cosine is put onto the shaft of the servomultiplier, and R is put across the potentiometer, then the error standard deviation will be proportional to R as postulated earlier. This fact may be stated as

$$\sigma_{\epsilon} = \frac{R}{R_m}\,\sigma_{\epsilon F} \quad \text{where} \quad \sigma_{\epsilon F}$$

is the standard deviation of errors at full-scale, i.e., when $R = R_m$. If the other multiplications are treated similarly, then it may be seen that

$$\sigma_{\dot{\ell}} \leq \frac{\sqrt{2}}{100}\, W\, \sigma_{\epsilon F}\,. \tag{98}$$

This result is somewhat more favorable than that for the uncorrelated case. The drift rate tends to be proportional to the existing rate of rotation rather than the full-scale rate. There would be little difference between the two if the rate were near the maximum most of the time, but this is not usually the case. In fact in many simulations, the rotation rate is small most of the time, the maximum rates being required only once or a few times.

It is now of interest to investigate the effect of the correction scheme on this length drift. We consider first the case of uncorrelated errors. If the complete Equations (92) are substituted in Equation (94), the result is

$$\begin{aligned}\dot{\ell} = K_1(1 - \vec{i}\cdot\vec{i}) + \Big\{ & a_{11}R_m \frac{\epsilon_{21R}}{100}\, a_{11}Q_m \frac{\epsilon_{31Q}}{100} + a_{21}P_m \frac{\epsilon_{31P}}{100} \\ & - a_{21}R_m \frac{\epsilon_{11R}}{100} + a_{31}Q_m \frac{\epsilon_{11Q}}{100} - a_{31}P_m \frac{\epsilon_{21P}}{100}\Big\} \\ & + \frac{K_1}{100C_1}\left[a_{11}\epsilon_{11c} + a_{21}\epsilon_{21c} + a_{31}\epsilon_{31c}\right].\end{aligned} \tag{99}$$

Consider the first term in this equation. Observe that $\vec{i}\cdot\vec{i}$ is simply ℓ^2. If it is assumed that $\ell = 1.0 + \Delta\ell$, then $\ell^2 \simeq 1.0 + 2\Delta\ell$, so that the first term becomes $-K_1 2\Delta\ell$. The second term is the same as the length drift rate of Equation (95). The last term is the drift rate due to the correction mechanism itself. For the present, it is assumed that C_1 is large enough that this last term is negligible with respect to the first two. We will return to this point later. With this assumption, then, Equation (99) becomes

$$\begin{aligned}\dot{\ell} = -K_1 2\Delta\ell + \Big\{ & a_{11}R_m \frac{\epsilon_{21R}}{100} - a_{11}Q_m \frac{\epsilon_{31Q}}{100} + a_{21}P_m \frac{\epsilon_{31P}}{100} - a_{21}R_m \frac{\epsilon_{11R}}{100} \\ & + a_{31}Q_m \frac{\epsilon_{11Q}}{100} - a_{31}P_m \frac{\epsilon_{21P}}{100}\Big\}\end{aligned} \tag{100}$$

Combining Equations (102) and (103), the limits on K_1 are

$$\frac{W_m \sigma_\epsilon}{100\sqrt{2}\,\sigma_{\Delta f}} \leq K_1 \leq \frac{W_m C_1}{4} .$$

There are several differences for correlated errors. When $P = Q = R = 0$, then all errors will be zero, provided the velocity components have been put across the multiplying potentiometers as suggested above. Under these circumstances, the drift is determined by integrator drift, and is at least an order of magnitude lower than the drifts normally arising from multiplier errors. If, however the length tolerance is to be met under the largest allowable rates of rotation, then the lower bound on K_1 is the same as that of Equation (102). The upper bound will disappear if the error signal is put across the potentiometer in the (multiplications. In this case, the error arising in the multiplier would be o the order of ϵ^2 rather than of the order of ϵ. The gain would not have an upper bound then, except for the fact that integrators tend to drift somewhat faster the higher the gains preceeding them.

Next the angular drift rate will be considered. Even if the coordinate system is kept orthogonal, there is still the tendency to drift in orientation, which comes principally from static errors in the multipliers. The components P, Q and R express the angular velocity in the moving system. If P_o, Q_o and R_o are the components in the fixed system, it may be shown that

$$\begin{aligned}
R_o &= a_{11}\dot{a}_{12} + a_{21}\dot{a}_{22} + a_{31}\dot{a}_{32} ; \\
-R_o &= a_{12}\dot{a}_{11} + a_{22}\dot{a}_{21} + a_{32}\dot{a}_{31} , \\
P_o &= a_{12}\dot{a}_{13} + a_{22}\dot{a}_{23} + a_{32}\dot{a}_{33} , \\
-P_o &= a_{13}\dot{a}_{12} + a_{23}\dot{a}_{22} + a_{33}\dot{a}_{32} , \\
Q_o &= a_{13}\dot{a}_{11} + a_{23}\dot{a}_{21} + a_{33}\dot{a}_{31} , \\
-Q_o &= a_{11}\dot{a}_{13} + a_{21}\dot{a}_{23} + a_{31}\dot{a}_{33} .
\end{aligned} \qquad (104)$$

This is a first order differential equation for Δl, the right-hand side or forcing term being a combination of errors. As the errors change during the course of a run, then the length error Δl will change also, but for simplicity consider a static case. If the coordinate system is not rotating, then all direction cosines are fixed, and the right side of Equation (100) becomes a constant. Let us call it u. This constant will have different values depending on the set of multipliers used, and in fact the standard deviation of the values it can assume is given by Equation (97). In the steady-state, $\Delta \dot{l}$ must be zero, and from Equation (100)

$$2K_1 \Delta l = u$$

The deviation in Δl will then be given by

$$\sigma_{\Delta l} = \frac{1}{2K_1}\sigma_u = \frac{\sigma_\epsilon}{K_1 100\sqrt{2}} W_m . \qquad (101)$$

σ_ϵ is fixed by the nature of the multipliers, W_m by the scaling requirements. K_1 may be chosen, however, to make $\sigma_{\Delta l}$ as small as desired. Conversely, the required value of K_1 is

$$K_1 \geq \frac{W_m}{100\sqrt{2}} \frac{\sigma_\epsilon}{\sigma_{\Delta l}} \qquad (102)$$

It was mentioned that the third term of Equation (99) should be negligible with respect to the first two. The reason for this is not clear from what has been presented thus far, but it will be seen later when angular drift is considered. Angular drift is determined by a similar equation, and it seems unreasonable to allow the length correction circuit to contribute to the drift in angle when it can be avoided, as will now be shown, by proper distribution of gains. The third term of Equation (99) will have the standard deviation $\frac{K_1}{100C_1}\sigma_\epsilon$, while the standard deviation of the second is $\frac{\sigma_\epsilon}{100} W_m$. If it is required that the third term be less than one quarter of the second, then

$$\frac{K_1\sigma_\epsilon}{100C_1} < \frac{\sigma_\epsilon W_m}{400} . \qquad (103)$$

$$\frac{K_1}{C_1} < \frac{W_m}{4}$$

which determines C_1, once K_1 is chosen. This establishes the upper limit on K_1.

Now substituting Equations (92) into the third of these gives

$$-R_o = -a_{13}P - a_{23}Q - a_{33}R + \left\{ a_{12}R_m \frac{\epsilon_{21R}}{100} - a_{12}Q_m \frac{\epsilon_{31Q}}{100} + a_{22}P_m \frac{\epsilon_{31P}}{100} - a_{22}R_m \frac{\epsilon_{11R}}{100} - a_{32}Q_m \frac{\epsilon_{11Q}}{100} + a_{32}P_m \frac{\epsilon_{21P}}{100} \right\}$$

$$\frac{K_1}{100C_1}\left[a_{12}\,\epsilon_{11c} + a_{22}\,\epsilon_{21c} + a_{32}\,\epsilon_{31c} \right]. \tag{105}$$

The first three terms represent the transformation of the velocity vector into the fixed axis system. In other words, they represent the "correct" value of R_o. The remaining terms represent the error in R_o, or the component of the drift velocity vector along the unprimed Z axis. Combination of the errors is similar to that for drift in length. If it is required that the last term of Equation (105) be negligible with respect to the others, then the standard deviation of drift rate is given by

$$\sigma_{\Delta R_o} \leq W_m \frac{\sigma_\epsilon \sqrt{2}}{100}$$

$$\sigma_{\Delta R_o} \leq W \frac{\sigma_{\epsilon F} \sqrt{2}}{100} \tag{106}$$

for the uncorrelated and correlated error cases respectively. Thus it appears that the drift rate will be a constant fraction of full scale, for uncorrelated errors, and will be a constant fraction of the angular velocity for correlated errors. The conditions on K_1 and C_1 are the same as developed earlier, namely

$$\frac{\sigma_\epsilon}{100\sqrt{2}\,\Delta l_1} < \frac{K_1}{W_m} < \frac{C_1}{4} \tag{107}$$

The drifts in P and Q are substantially the same provided gains are chosen such as to make the drift contribution of the correction circuits negligible. The case of K_2 and C_2 is substantially the same as for K_1 and C_1, and they should be chosen in the same way.

Analyses of drifts in P and Q are done is much the same way and lead to similar results.

B. Quaternion Error Analysis

The quaternion simulation may be handled in much the same way. The quaternion components are not all independent, and we may make use of the relationship,

$$e_1^{\,2} + e_2^{\,2} + e_3^{\,2} + e_4^{\,4} = 1 , \tag{32}$$

in the same way that $a_{11}^{\,2} + a_{21}^{\,2} + a_{31}^{\,2} = 1$ was used to maintain the length of the unit vector. The equations to be solved are

$$\begin{aligned}
2\dot{e}_1 &= -e_4P - e_3Q - e_2R + K\phi e_1 ,\\
2\dot{e}_2 &= -e_3P + e_4Q + e_1R + K\phi e_2 ,\\
2\dot{e}_3 &= e_2P + e_1Q - e_4R + K\phi e_3 ,\\
2\dot{e}_4 &= e_1P - e_2Q + e_3R + K\phi e_4 ,\\
\phi &= 1 - e_1^{\,2} - e_2^{\,2} - e_3^{\,2} - e_4^{\,2} .
\end{aligned} \tag{108}$$

The bandpass requirement of this method is only one half as severe as that for direction cosines. This may be seen in several ways. Consider the equations above with Q constant and $P = R = K = 0$.

$$\begin{aligned}
\dot{e}_1 &= -e_3\frac{Q}{2} ,\\
\dot{e}_2 &= +e_4\frac{Q}{2} ,\\
\dot{e}_3 &= +e_1\frac{Q}{2} ,\\
\dot{e}_4 &= -e_2\frac{Q}{2}
\end{aligned} \tag{109}$$

For initial conditions, we assume $e_1 = 1$, $e_2 = e_3 = e_4 = 0$. Under these conditions, e_2 and e_4 will remain zero and the equations become

$$\begin{aligned}
\dot{e}_1 &= -e_3\frac{Q}{2} ,\\
\dot{e}_3 &= e_1\frac{Q}{2} ,
\end{aligned} \tag{110}$$

These are identical with those treated in the bandpass study of the preceeding section except that the frequency is reduced by one half. The same conclusion can be seen from the definition of e_1. From Equation (33), $e_1 = \cos \mu/2$, so while u is completing a full rotation, $\mu/2$ only moves through 180°. This means that the servos which are driven by the e's only move with half the speed of those driven by the a's, for a given rate of rotation of the coordinate system. Consequently, for a given accuracy, twice as much phase shift may be allowed.

In order to analyze the effect of static multiplier errors, it is again necessary to postulate some particular scaling. The simplified diagram is shown in Figure 2. The equipment necessary for determining the direction cosines has been included, as they will always be needed. The basic quaternion component computation requires 20 multipliers and four integrators. Conversion to direction cosines requires another six multipliers, for a total of 26 multipliers and 4 integrators, against 36 multipliers and 6 integrators for direction cosines. Notation for individual multiplier errors is similar to that applied earlier. The voltage error in the multiplier which multiplies R and e_1, for instance, is called ϵ_{1R}, and so on. From this figure, it may be seen that the equations being solved are:

$$
\begin{aligned}
2\dot{e}_1 &= -e_4P - e_3Q - e_2R + K\phi e_1 - \epsilon_{4P}\frac{P_m}{100} - Q_m\frac{\epsilon_{3Q}}{100} - R_m\frac{\epsilon_{2R}}{100} + K_c\frac{\epsilon_{1c}}{100} \\
2\dot{e}_2 &= -e_3P + e_4Q + e_1R + K\phi e_2 - P_m\frac{\epsilon_{3P}}{100} + Q_m\frac{\epsilon_{4Q}}{100} + R_m\frac{\epsilon_{1R}}{100} + K_c\frac{\epsilon_{2c}}{100} \\
2\dot{e}_3 &= +e_2P + e_1Q - e_4R + K\phi e_3 + P_m\frac{\epsilon_{2P}}{100} + Q_m\frac{\epsilon_{1Q}}{100} - R_m\frac{\epsilon_{4R}}{100} + K_c\frac{\epsilon_{3c}}{100} \\
2\dot{e}_4 &= +e_1P - e_2Q + e_3R + K\phi e_4 + P_m\frac{\epsilon_{1P}}{100} - Q_m\frac{\epsilon_{2Q}}{100} + R_m\frac{\epsilon_{3R}}{100} + K_c\frac{\epsilon_{4c}}{100}
\end{aligned}
\qquad (111)
$$

Consider first the effect of errors on the length of the quaternion. The length is given by $\ell^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2$. Differentiating with respect to time gives $\ell\dot{\ell} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$. Since the length is not to be allowed to vary much from unity, the drift rate may be approximated by $\dot{\ell} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$. Substituting Equations (111) into this relation and performing the appropriate reductions gives:

$$2\dot{\ell} = K\phi + \frac{1}{100}\{-e_1 P_m \epsilon_{4P} - e_1 Q_m \epsilon_{3Q} - e_1 R_m \epsilon_{2R} - e_2 P_m \epsilon_{3P}$$

$$+ e_2 Q_m \epsilon_{4Q} + e_2 R_m \epsilon_{1R} + e_3 P_m \epsilon_{2P} + e_3 Q_m \epsilon_{1Q}$$

$$- e_3 R_m \epsilon_{4R} + e_4 P_m \epsilon_{1P} - e_4 Q_m \epsilon_{2Q} + e_4 R_m \epsilon_{3R}\}$$

$$+ \frac{K}{100C}\{ e_1 \epsilon_{1c} + e_2 \epsilon_{2c} + e_3 \epsilon_{3c} + e_4 \epsilon_{4c} \} . \qquad (112)$$

If $\ell = 1 + \Delta\ell$, then the first term is simply $-2K\Delta\ell$. Thus Equation (112) becomes

$$\dot{\Delta\ell} + K\Delta\ell = \frac{1}{200}\{ -e_1 P_m \epsilon_{4P} - e_1 Q_m \epsilon_{3Q} - e_1 R_m \epsilon_{2R} - e_2 P_m \epsilon_{3P}$$

$$+ e_2 Q_m \epsilon_{4Q} + e_2 R_m \epsilon_{1R} + e_3 P_m \epsilon_{2P} + e_3 Q_m \epsilon_{1Q}$$

$$- e_3 R_m \epsilon_{4R} + e_4 P_m \epsilon_{1P} - e_4 Q_m \epsilon_{2Q} + e_4 R_m \epsilon_{3R}\}$$

$$+ \frac{K}{200C}\{ e_1 \epsilon_{1c} + e_2 \epsilon_{2c} + e_3 \epsilon_{3c} + e_4 \epsilon_{4c} \} \qquad (113)$$

which is analogous with Equation (100).

As in the preceeding section, we assume the second term on the right of Equation (113) to be negligible with respect to the first. The variance of the first term is

$$\frac{\sigma_\epsilon}{200} W_m .$$

Thus, for the steady-state,

$$\sigma_{\Delta\ell} = \frac{\sigma_\epsilon}{200} \frac{W_m}{K} \qquad (114)$$

$$K \geq \frac{\sigma_\epsilon}{\sigma_{\Delta\ell}} \frac{W_m}{200}$$

where $\sigma_{\Delta\ell}$ is the standard deviation in the length error which will be allowed. The standard deviation of the second terms of Equation (113) is $\frac{K}{200C} \sigma_\epsilon$. The requirement that the second term be no more than one quarter the first gives

$$K < \frac{W_m}{4} C$$

Combining the two gives almost the same conditions on K as were obtained for the direction cosine method.

$$\frac{\sigma_\epsilon}{200}\,\frac{W_m}{\sigma_{\Delta l}} < K < \frac{W_m}{4}\,C\,. \tag{115}$$

The foregoing is for uncorrelated errors. The changes in the correlated error case would be the same as in the direction cosine analysis: the upper bound on K disappears and the lower bound is the same as in the uncorrelated case.

Next the effect of multiplier errors on the angular drift must be evaluated. Equation (79) gives the rotation rate components P, Q, and R in terms of the rates of change of the quaternion components. The apparent value of P, which we we call P_a is, the $P_a = 2(-e_4\dot{e}_1 - e_3\dot{e}_2 + e_2\dot{e}_3 + e_1\dot{e}_4)$. If Equations (111) are substituted into this expression, the result is

$$\begin{aligned} P_a = (e_1^{\,2} + e_2^{\,2} + e_3^{\,2} + e_4^{\,2})\,P &+ \frac{1}{100}\{e_4(P_m\,\epsilon_{4P} + Q_m\,\epsilon_{3Q} + R_m\,\epsilon_{2R}) \\ &+ e_3(P_m\,\epsilon_{3P} - Q_m\,\epsilon_{4Q} - R_m\,\epsilon_{1R}) + e_2(P_m\,\epsilon_{2P} + Q_m\,\epsilon_{1Q} - R_m\,\epsilon_{4R}) \\ &+ e_1(P_m\,\epsilon_{1P} - Q_m\,\epsilon_{2Q} + R_m\,\epsilon_{3R})\} + \frac{K}{100C}\{-e_4\,\epsilon_{1c} - e_3\epsilon_{2c} + e_2\,\epsilon_{3c} + e_1\epsilon_{4c}\}\,. \end{aligned} \tag{116}$$

The first term is equivalent to $(1 + 2\Delta l)P$ so the error in P, $(P_a - P)$ is

$$\begin{aligned} \Delta P = P_a - P = 2\Delta l\,P &+ \frac{1}{100}\{e_4(P_m\epsilon_{4P} + Q_m\,\epsilon_{3Q} + R_m\,\epsilon_{2R}) \\ &+ e_3(P_m\,\epsilon_{3P} - Q_m\,\epsilon_{4Q} - R_m\,\epsilon_{1R}) + e_2\,(P_m\,\epsilon_{2P} + Q_m\,\epsilon_{1Q} - R_m\,\epsilon_{4R}) \\ &+ e_1(P_m\,\epsilon_{1P} - Q_m\,\epsilon_{2Q} + R_m\,\epsilon_{3R})\} + \frac{K}{100C}\{-e_4\epsilon_{1c} - e_3\epsilon_{2c} + e_2\epsilon_{3c} + e_1\,\epsilon_{4c}\}. \end{aligned} \tag{117}$$

$2\Delta l$ is related to the choice of K. From Equation (114) it may be seen that the standard deviation in Δl is given by $\sigma_{\Delta l} = \frac{\sigma_\epsilon}{200K}\,W_m$. As before, we require the last term of Equation (117) to be negligible. It remains to determine the variance of the second term. By taking the sum of variances of the individual terms, it is possible to show that the standard deviation of the second term is $\frac{1}{100}\,W_m\,\sigma_\epsilon$.

Thus the standard deviation in ΔP is

$$\sigma_{\Delta P} = \frac{\sigma_\epsilon}{100} W_m \sqrt{1 + \frac{P^2}{\sqrt{2}K^2}} \tag{118}$$

This is comparable to the value obtained for direction cosines if K is made reasonably large.

Exactly analogous results are obtained for errors in the other two axes.

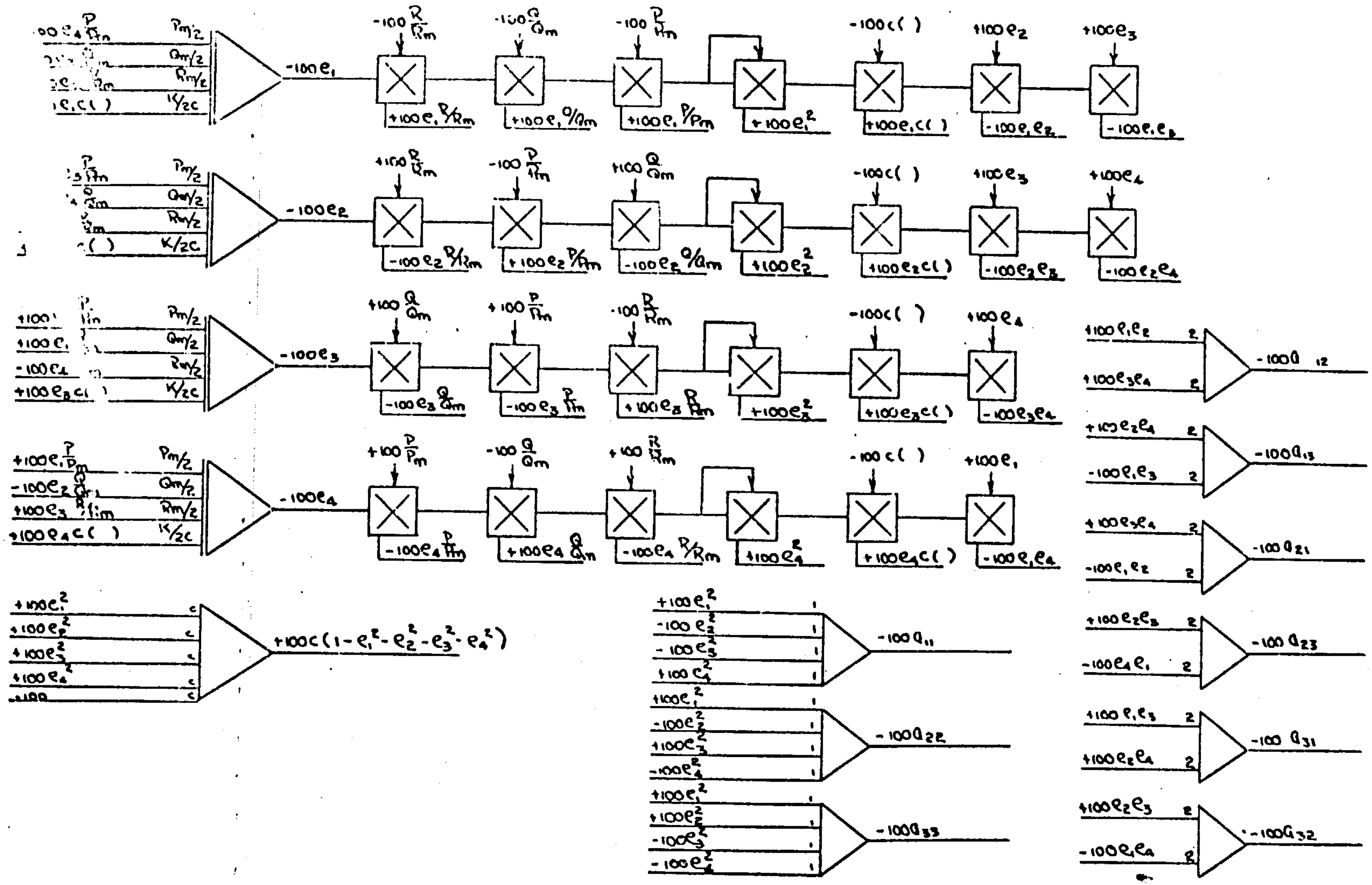

-100e₁
-100e₂
-100e₃
-100e₄
+100c(1-e₁²-e₂²-e₃²-e₄²)
-100a₁₁
-100a₂₂
-100a₃₃
-100a₁₂
-100a₁₃
-100a₂₁
-100a₂₃
-100a₃₁
-100a₃₂

Figure 2

SECTION VII

SIMULATOR RESULTS

The next step in investigation of quaternion coordinate conversion was solution of the equations on an analog computer. The direction cosine method was similarly investigated, and an attempt was made to make the conditions of the two simulations as nearly alike as their inherent differences would permit. Both were done on REAC Series 100 equipment. Servomultipliers equipped with potentiometers of 0.05 percent linearity were used throughout. The multipliers were not specially calibrated for these simulations, so their adjustment was consistent with normal practice in the Analog Computation Branch, Aeronautical Research Laboratory. Of course, correct pot loading was used in all cases. In both simulations, a maximum rotational velocity of 0.5 radians/sec was employed though scaling was such that $P_m = Q_m = R_m = 1.0$ and both directions cosines and the quaternion components were scaled so that 50 volts (out of 100 full scale) represented the extreme possible excursion of the variable. This was done to get away from possible end effects on servomultiplier potentiometers.

Adequate checking of a coordinate conversion is a problem in itself, and while it is not claimed that the method adopted here meets all requirements, it seems sufficient, and no better method presented itself. With the types of coordinate conversion considered here, there are two things principally to be checked: the action of the "orthogonalization" or correction mechanisms, and the rotation drift rate. The first may be checked simply by monitoring the error quantities which are used for correction. The drift rate is not so easily checked. In most cases the drift rate will be very small compared with the rate of rotation. An exception to this is the case when an angular velocity of zero is desired. Any shift which takes place under these conditions is readily detected. To check the drift while rotating, the following procedure was adopted: a single input of $P = 0.500$ rad/sec (Q and R zero) was applied to the equations for a period of approximately 125.66 seconds. This is enough time for ten complete revolutions at this frequency. The transformation matrix should be the same at the end of this period as it was at the beginning, except for drift. At the start of each run, the transformation was the identity transformation, so the matrix existing at the end of the run has a simple interpretation. Because of the difficulty of controlling the length of run with sufficient accuracy, several runs were made using the same conditions, and the time of each run was recorded. It was then possible to plot the rotation angle as a function of run time, and by interpolation, to determine the drift angle existing at exactly 125.66 seconds. The same process was repeated for $P = R = 0$, $Q = 0.5$ and $P = Q = 0$, $R = 0.5$, thus giving rotation about each of the three axes singly.

A. The Quaternion Method

It was mentioned earlier that the problem was scaled for 50 volts maximum on the e's rather than 100 volts. For this case, Equation (118) becomes

$$\sigma_{\Delta P} = \frac{\sigma_\epsilon}{50} W_m \sqrt{1 + \frac{P^2}{\sqrt{2}\, K^2}} \,. \tag{119}$$

The value of P was 0.5, K was 2.0, and servomultipliers were used, so for correlated errors, this expression becomes

$$\sigma_{\Delta P} = \frac{\sigma'_\epsilon}{100} (1.03). \tag{120}$$

It is shown in Appendix C that σ'_ϵ is 0.05 volts. Therefore, the standard deviation in P is 5.15×10^{-4} radians/second. In 125 seconds, this would amount to 0.0644 radians or about 3.7 degrees. The standard deviation of the drift angle after 125 seconds, then should be about 3.7 degrees. The drift angle was determined only three times, once each for P, Q, and R inputs. Three results constitute an insufficient number of samples for statistical significance. In order to get the number of results required, it would be necessary to do the entire setup many times using different pots in different permutations. It was not felt that the improved confidence in the error analysis would justify the immense labor of this procedure. The results of the three determinations which were made are not inconsistent with the theoretical errors found.

At the end of each run, the transformation matrix existing is very nearly the identity transformation. In order to interpret this final matrix, it is convenient to make use of Equation (164) of Appendix A,

$$(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \Delta\mu \cos\gamma & -\Delta\mu \cos\beta \\ -\Delta\mu \cos\gamma & 0 & \Delta\mu \cos\alpha \\ \Delta\mu \cos\beta & -\Delta\mu \cos\alpha & 0 \end{pmatrix}, \tag{121}$$

... drift angle, and α, β, and γ are the angles between the drift axis and the x, y, and z axes respectively. Thus, after the run, a_{11}, a_{22} and a_{33} should be unity. All the other direction cosines should be small, but may differ from zero. If they are small, the following relationships should hold:

$$a_{12} = -a_{21}; \qquad a_{13} = -a_{31}, \qquad a_{23} = -a_{32}.$$

The procedure was as follows. At the start of each run, initial conditions of the four integrators were set to correspond to the identity transformation, that is $e_1 = 1$, $e_2 = 0$, $e_3 = 0$, $e_4 = 0$. Then the computer was put into the "Operate" condition. This switching also started a timer driven by a synchronous motor from the 60 cycle line voltage. An integrator and biased relay were used to terminate the run, and this termination stopped the timer. The time of the run could then be read directly. At termination, the computer was put into "Hold" and voltages corresponding to the nine direction cosines were read to the nearest 10 millivolts with a digital voltmeter. The results obtained are included in Table I. In this table also are included the drift angle, and direction cosines of the drift axes for each run made.

TABLE I

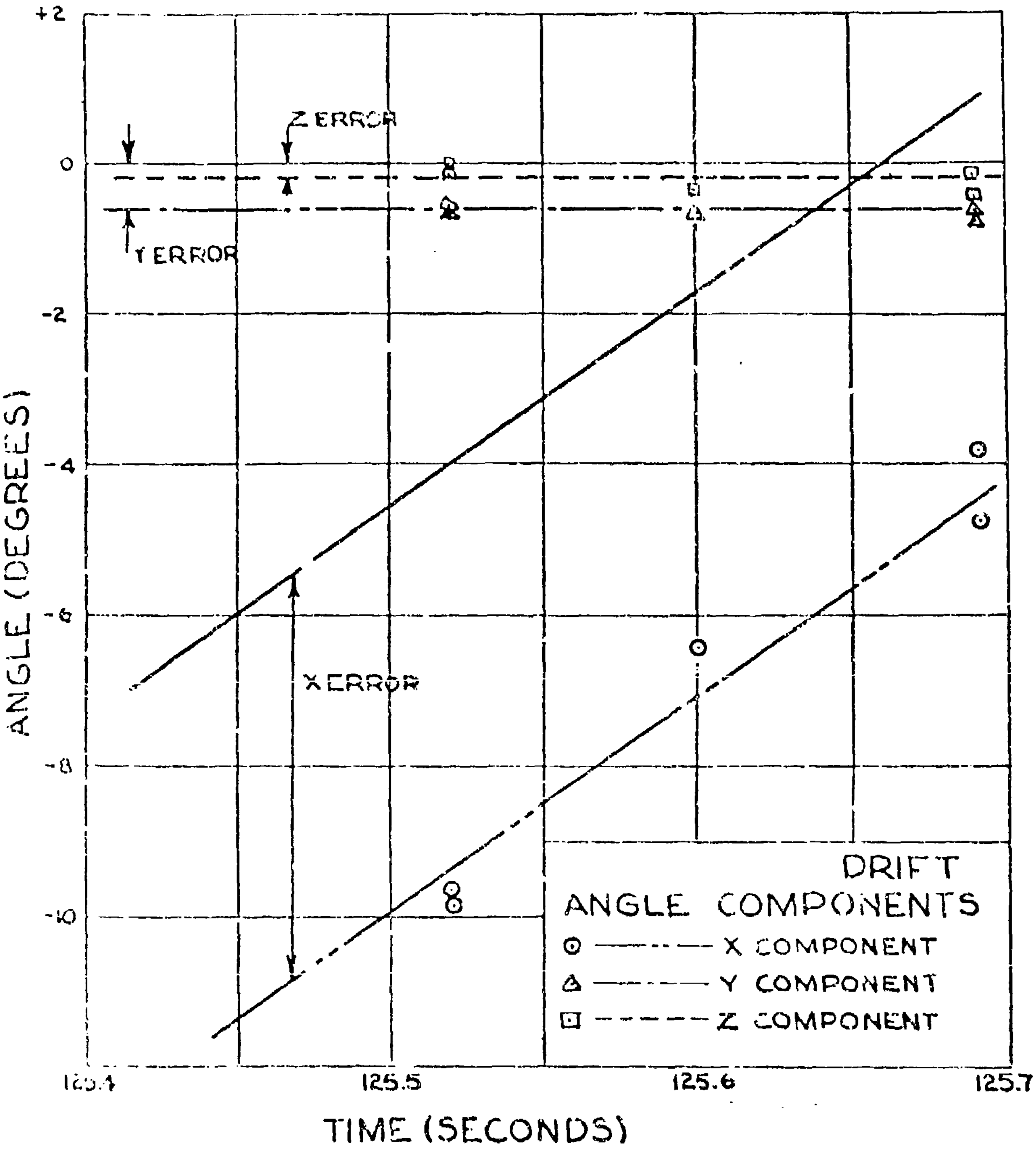
DRIFT RUNS, QUATERNION METHOD

		Time (Sec)	$+50a_{11}$	$+50a_{22}$	$+50a_{33}$	$+50a_{12}$	$+50a_{21}$	$+50a_{31}$	$+50a_{13}$	$+50a_{23}$	$+50a_{32}$	Drift (o)
		125.52	+49.93	+49.19	+49.08	-00.02	+00.00	+00.54	-00.46	-08.57	+08.65	09.88
2		125.52	+49.91	+49.24	+49.13	-00.15	+00.12	+00.60	-00.50	-08.33	+08.44	09.64
3		125.69	+49.93	+49.85	+49.75	-00.37	+00.40	+00.57	-00.47	-03.31	+03.39	03.91
4	P	125.69	+49.94	+49.80	+49.69	-00.13	+00.14	+00.67	-00.59	-04.07	+04.22	04.80
5	P	125.60	+49.94	+49.64	+49.53	-00.29	+00.29	+00.63	-00.52	-05.58	+05.69	06.48
6	Q	125.50	+48.99	+49.92	+48.91	+01.10	-00.95	-09.55	+09.61	-00.53	+00.41	11.72
7	Q	125.48	+48.71	+49.93	+48.62	+01.15	-01.01	-11.00	+11.08	-00.34	+00.19	12.72
8	Q	125.45	+48.75	+49.92	+48.63	+00.38	-03.51	-10.89	+10.97	+00.96	-00.92	12.58
9	Q	125.52	+49.05	+49.90	+48.94	-00.86	+00.80	-09.38	+09.44	+00.86	-00.60	10.85
10	Q	125.55	+49.29	+49.92	+49.19	+01.03	-01.00	-07.91	+07.99	+00.31	-00.39	09.17
11	Q	125.79	+49.85	+49.93	+49.78	+00.85	-00.81	-02.55	+02.63	+00.61	-00.59	03.19
12	Q	125.86	+49.92	+49.94	+49.94	+00.82	-00.75	-00.29	+00.38	+00.63	-00.57	01.20
13	R	125.82	+49.92	+49.96	+49.85	+00.92	-00.85	+00.74	-00.66	+00.12	-00.03	01.33
14	R	125.70	+49.90	+49.93	+49.86	-01.83	+01.90	+00.76	-00.69	+00.18	-00.12	02.31
15	R	125.56	+49.54	+49.57	+49.84	-06.19	+06.26	+00.74	-00.66	+00.10	-00.11	07.19
16	R	125.60	+49.79	+49.81	+49.85	-03.93	+04.00	+00.71	-00.63	+00.12	-00.10	04.62
17	R	125.59	+49.74	+49.77	+49.84	-04.47	+04.55	+00.76	-00.69	+00.09	-00.09	05.24
18	None	125.65	+49.93	+49.95	+49.85	+00.53	-00.46	+00.38	-00.30	+00.56	-00.49	00.92
19	None	125.69	+49.93	+49.95	+49.85	+00.53	-00.46	+00.39	-00.31	+00.56	-00.49	00.93
20	None	125.73	+49.94	+49.96	+49.85	+00.53	-00.46	+00.39	-00.31	+00.55	-00.48	00.92
21	None	125.38	+49.91	+49.91	+49.80	+00.13	-00.15	+00.00	+00.00	+00.13	-00.13	00.22
22	None	125.64	+49.91	-49.90	+49.78	+00.13	-00.15	+00.00	+00.00	+00.19	-00.18	00.27
23	None	125.60	+49.92	+49.92	+49.80	+00.12	-00.15	+00.00	+00.00	+00.22	-00.20	00.29
24	None	125.60	+49.91	+49.94	+49.83	+00.14	-00.16	+00.00	+00.00	+00.19	-00.18	00.28
25	None	600.00	+49.84	+49.84	+49.72	+00.87	-00.89	+00.36	-00.35	+00.72	-00.75	01.39
26	None	600.00	+49.89	+49.89	+49.79	+00.99	-01.01	+00.47	-00.45	+00.80	-00.83	01.57
27	None	0	+49.89	+49.92	+49.82	+00.01	+00.01	+00.00	+00.03	+00.14	-00.07	-

The first five runs, those made with P = 0.5, Q = R = 0, may be used to illustrate several points of interest. In all cases, the drift angle will be small, corresponding to an infinitesmal rotation. Since an infinitesmal rotation may be treated as a vector, we may take components of this vector along the three axes of the moving system; these components are simply the a_{23}, a_{31} and a_{12} of the final matrix. These are plotted as functions of time in Figure 3. For this case, perfect performance would dictate that the y and z components remain zero, and the x component increases at a rate of 28.65 deg/sec (0.5 rad/sec), passing through zero at 125.66 seconds. This latter is shown as the heavy solid curve of Figure 3. It may readily be seen how the results achieved vary from this simplification. The z error is about 0.2°, the y error is about 0.6° and the x error is about 5.35°. This illustrates the result that in every case the major portion of the drift is in the direction of rotation. This is to be expected when servomultipliers are used.

Figure 4 shows the total drift angle as a function of time for all three cases. Again the ideal curve is the heavy line. The P, Q and R cases show 5.2, 7.0 and 3.4 degrees respectively. It appears also, that all three curves lie significantly below the ideal. This was not predicted in the error analysis. The bias appeared in the direction cosine simulation as well, and a more detailed consideration will be given to it in the next part. For the present, it will simply be said that it was traced to the fact that the gains of the inverting amplifiers in the REAC are consistently slightly less than the indicated value. Aside from the bias, it seems that the dispersion agrees relatively well with the predicted standard deviation of 3.7 degrees. This does not mean much, however, in the presence of the bias. There is no reason to expect that the bias will be exactly the same in all three cases, so it cannot be determined what part of the dispersion is due to multiplier inaccuracy, and what is due to amplifier gain variations.

It was found possible in the direction cosine method to reduce the bias markedly by trimming the amplifier gains to exactly the desired value. It is felt, however, that the results should be presented as originally obtained, however, because of the avowed objective of showing what might be obtained in a practical simulation program. If it is desired, the drift could no doubt be reduced by detailed calibration and adjustment to about one tenth of that shown in Figure 4.



+2
0
-2
-4
-6
-8
-10
Z ERROR
Y ERROR
X ERROR
ANGLE (DEGREES)
DRIFT
ANGLE COMPONENTS
X COMPONENT
Y COMPONENT
Z COMPONENT
125.5
125.6
125.7
TIME (SECONDS)

Figure 5.

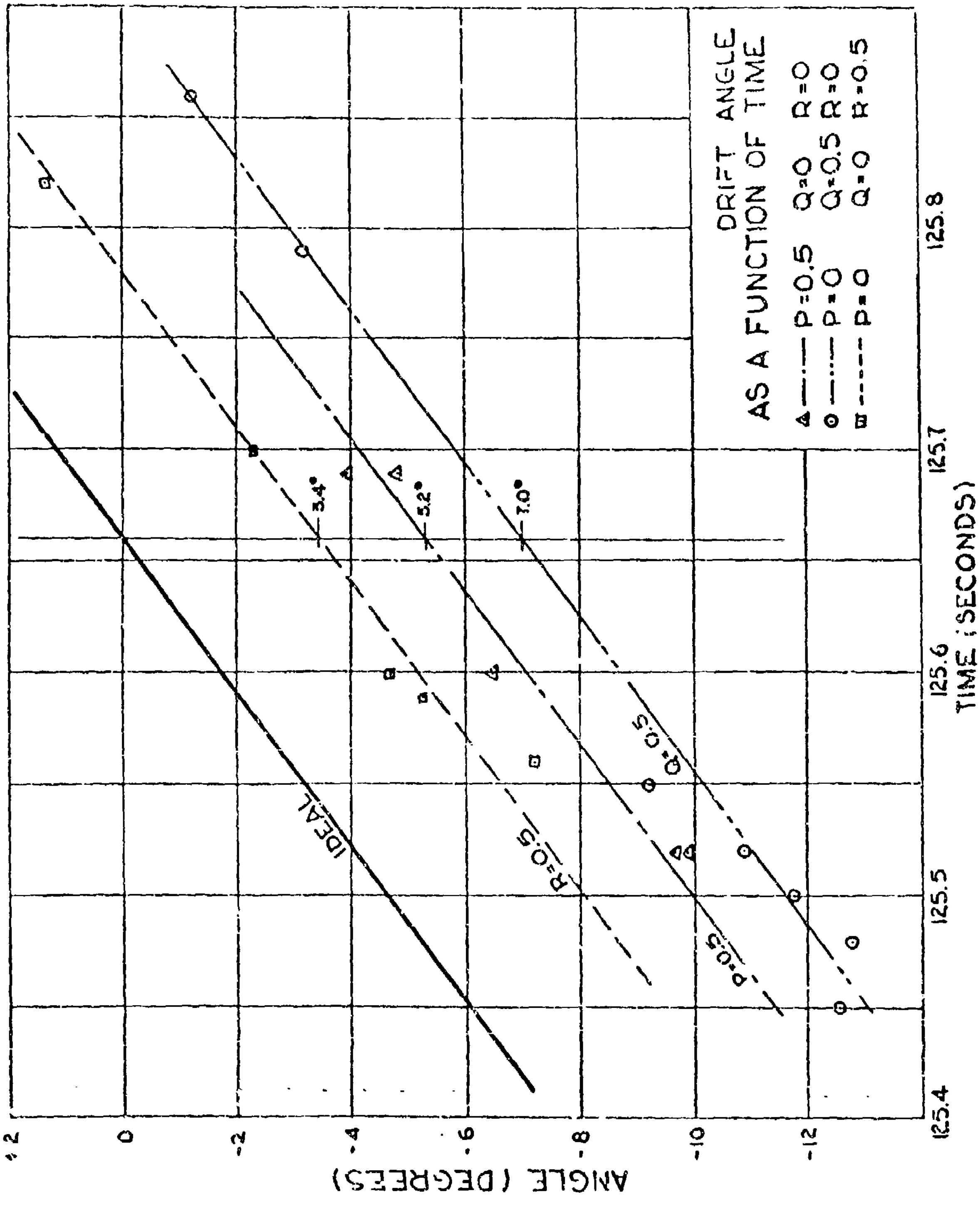

DRIFT ANGLE
AS A FUNCTION OF TIME
P=0.5 Q=0 R=0
P=0 Q=0.5 R=0
P=0 Q=0 R=0.5
IDEAL
R=0.5
P=0.5
Q=0.5
3.4°
5.2°
7.0°
TIME (SECONDS)
ANGLE (DEGREES)
125.4
125.5
125.6
125.7
125.8
2
0
-2
-4
-6
-8
-10
-12

Figure 4.

Figure 5 shows the action of the correction circuit as it eliminates an initial error.

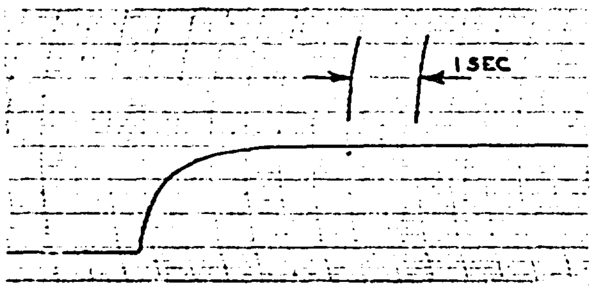


Figure 5. Correction Circuit Effect

Figure 6 shows a time history of part of one of the runs. It shows how the correction circuit maintains the length during a run. The indicated Δl is less than 0.1 per cent most of the time. This, of course, is merely the amplitude of the error signal. It does not necessarily mean that l is actually being held to this absolute precision, but rather that it is being held to the value of l which the computer shows to be unity.

The drift when P = Q = R = 0 is of considerable interest. In many simulations which make use of a coordinate conversion, the rate of rotation is small most of the time, reaching the peak values only occasionally. In such cases, the tendency to drift when the coordinate system should be standing still is of prime importance. For uncorrelated errors, there is no reason to expect that the drift will be much smaller in this condition than otherwise. It is true that the errors in electronic multipliers are somewhat smaller near zero than elsewhere, but the difference is not dramatic. This is particularly true when one of the inputs is large and the other is zero. This case will arise since, regardless of the orientation of the coordinate system, some of the e's will be large. For servomultipliers, on the other hand, the error tends to zero as the voltage across the multiplying pot goes to zero. This would seem to show that the drift would be exactly zero with no input rate. This is not correct, of course, because integrator drift is still present. Runs 18 through 27 of Table 1 were made under these conditions. Runs 18 through 20 were made on one day, and a drift rate of 7.25×10^{-3} deg/sec was observed. This amounts to about

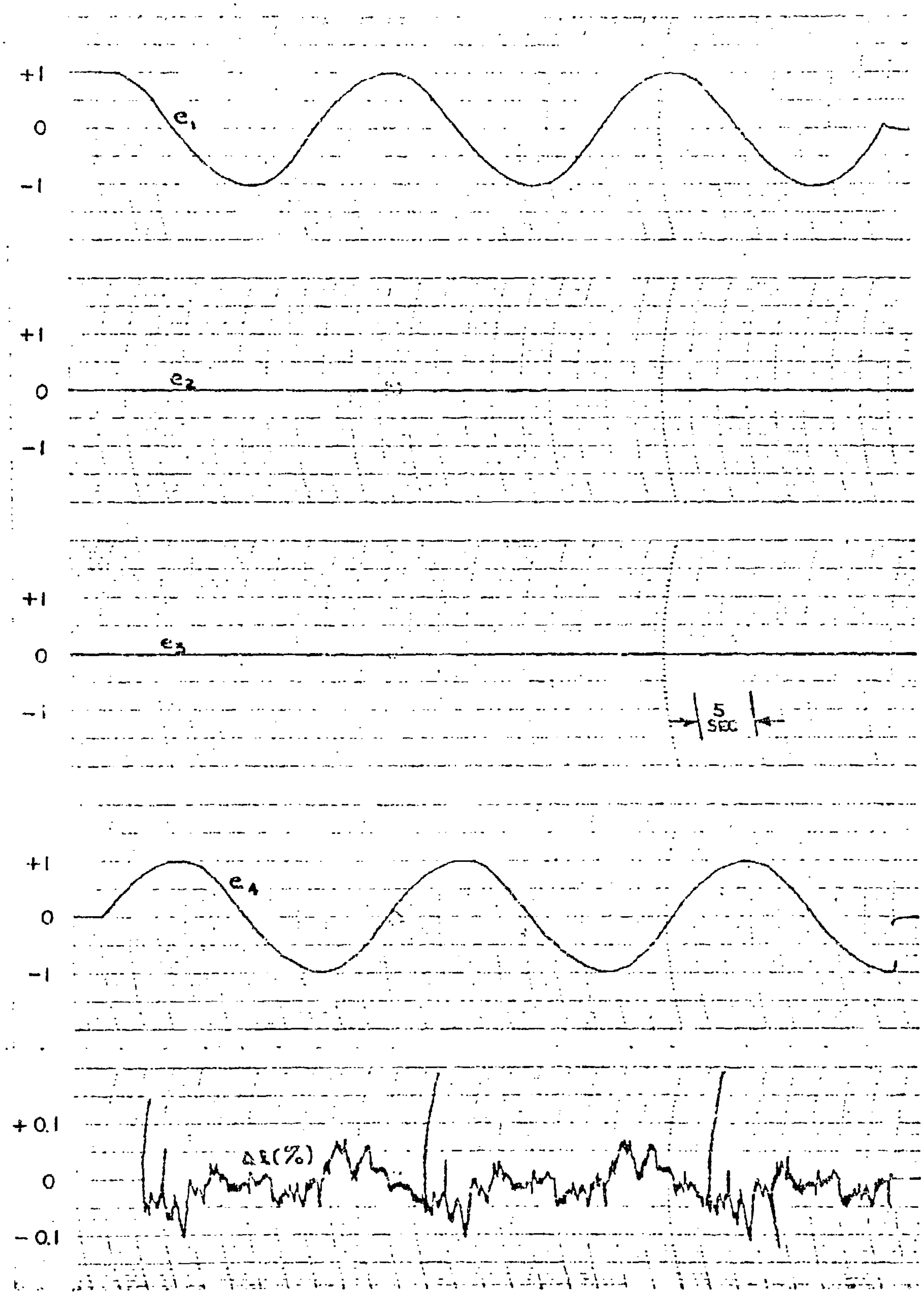


Figure 6. Time History for P = 0.5 Q = R = 0

0.0128 per cent of full scale (1 rad/sec). Runs 21 through 26 were made about a week after the preceeding set, after another problem had been on the computer in the meantime. From this second set, a drift rate of about 2.5×10^{-3} deg/sec was determined, an improvement by about the factor 3. Integrators had been balanced in both cases, so the only conclusion possible is that the zero-input drift is somewhat variable. It was found that the zero-input drift was proportional to the maximum rate for which the computer is scaled. The above results were taken for 1 rad/sec full scale, but some runs were made with 5 rad/sec full scale, and the drift was almost exactly five times as great.

While it does not appear utile to give the complete computer diagram, some remarks concerning the setup are in order. In order to accomplish the functions indicated in Figure 2, it was found necessary to use 36 summing and inverting amplifiers, 4 integrators, and 8 multiplying serves, each with three multiplying pots. One summing amplifier was used ahead of each integrator to do the summing. It now appears that this was not wise, because while the integrator gains proved quite accurate, the summing amplifier gains were less so, and introduced an error in angular rate. The simulation was not found to be critical or sensitive in any way, except that the zero-input drift varies somewhat from day to day. In all cases, however, it was quite acceptable.

B. The Direction Cosine Method

The direction cosine simulation was done under as nearly the same conditions as possible. The same rate of rotation (0.5 rad/sec) and the same full-scale rate were made and the same length of time was used. It was found necessary to time the runs with a Berkley counter, rather than the syncrhonous clock used in the quaternion case because of the failure of the latter unit at the start of the cosine runs. Such checks as could be made showed no significant difference in the timers. To do the operations of Figure 1, 31 inverting and summing amplifiers, 6 integrators and 12 servomultipliers were required. Table 2 shows the results obtained from reading the direction cosines at the end of each run. Again the rotation angle was plotted against time for each of the three inputs. These plots are given in Figure 7. Again it appears that the three curves show something like the predicted dispersion, but a large bias also. The bias in this

TABLE II

DRIFT RUNS, DIRECTION COSINE METHODS

	Input	Time (Sec)	$+50a_{11}$	$+50a_{22}$	$+50a$	$+50a_{12}$	$+50a_{21}$	$+50a_{31}$	$+50a_{13}$	$+50a_{23}$	$+50a_{32}$
1	P	125.45	+49.93	+47.50	+47.46	-00.04	+00.54	+01.76	-01.85	-15.48	+15.40
2	P	125.97	+49.94	+49.88	+49.92	+01.84	-01.86	-01.11	+00.00	-02.38	+02.33
3	P	125.77	+49.95	+49.14	+49.17	+01.70	-01.82	+00.40	-00.73	-08.83	+08.79
4	P	125.76	+49.94	+49.36	+49.38	+01.84	-01.81	-00.37	+00.08	-07.60	+07.57
5	P	125.77	+49.95	+49.07	+49.10	+01.80	-01.87	+00.05	-00.40	-09.25	+09.19
6	Q	125.59	+49.33	+50.02	+49.37	+00.09	-00.00	-08.37	+08.34	-08.69	[illegible]0.65
7	Q	125.66	+49.63	+50.02	+49.65	+00.00	-00.01	-06.26	+06.23	+00.05	-00.10
8	Q	125.77	+49.62	+50.02	+49.66	+00.11	-00.01	-06.32	+06.29	-00.88	-00.86
9	Q	125.73	+49.83	+50.00	+49.84	+00.07	+00.06	-04.45	+04.42	-01.00	+00.97
10	Q	125.73	+49.71	+50.03	+49.75	+00.01	-00.01	-05.49	+05.45	-00.23	+00.20
11	R	125.69	+49.16	+49.17	+50.00	-09.22	+09.13	+00.26	-00.29	+00.02	-00.11
12	R	125.70	+48.79	+48.82	+50.00	-10.95	+10.85	+00.26	-00.30	+00.01	-00.11
13	R	125.68	+49.13	+49.15	+50.0[illegible]	-09.33	+09.24	+00.27	-00.30	+00.02	-00.10
14	R	125.69	+49.17	+49.19	+50.0[illegible]	-09.14	+09.05	+00.26	-00.30	+00.00	-00.10
15	R	125.67	+49.01	+49.05	+50.02	-09.94	+09.84	+00.27	-00.30	+00.01	-00.10
16	None	125.54	+50.01	+50.04	+50.04	-00.11	+00.06	+00.24	-00.31	+00.09	-00.11
17	None	125.49	+49.98	+50.02	+50.0[illegible]	-00.12	+00.0[illegible]	+00.29	-00.31	+00.09	-00.13
18	None	125.49	+49.98	+50.02	+49.90	-00.09	+00.03	+00.26	-00.29	+00.09	-00.13
19	None	125.55	+49.97	+50.03	+50.0[illegible]	-00.03	+00.01	+00.30	-00.32	+00.07	-00.11
20	None	125.56	+49.98	+50.02	+50.0[illegible]	-00.06	+00.01	+00.30	-00.32	+00.09	-00.11

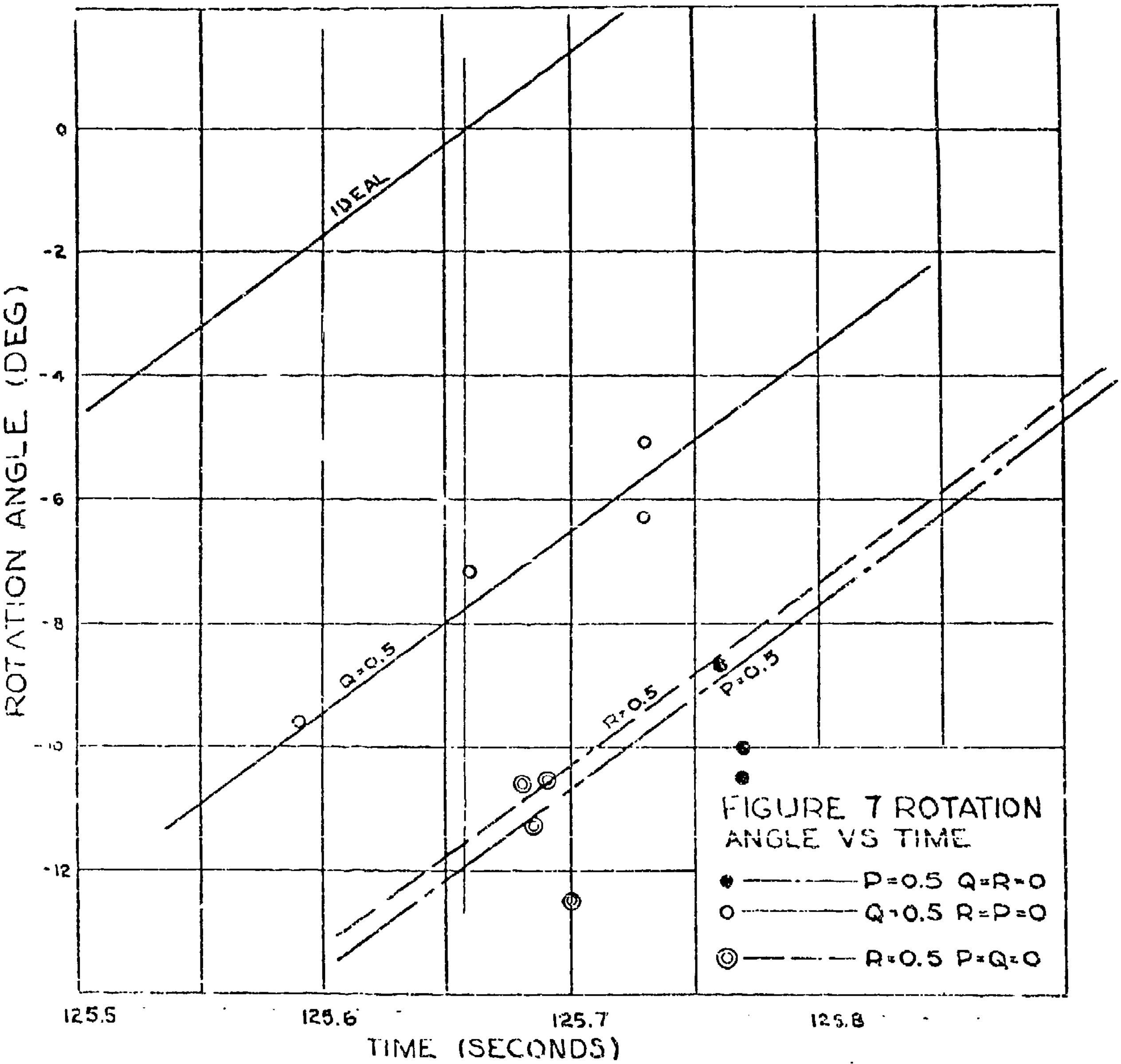


FIGURE 7 ROTATION ANGLE VS TIME

case is nearly twice that found for the quaternions. The cause proved somewhat difficult to locate, and it is constructive to consider some of the checks that were made in the process of looking for it.

This bias amounts to an erroneous rate of rotation. The coordinate system is rotating a trifle too slowly. In the example considered here, the system has completed about 10 revolutions, turning through a total angle of nearly 3600 degrees. At the end of this time, it is in error by some 10 degrees or so. This is one part of 360, surely not a large error, yet it is the predominant one, being nearly three times as large as the error due to multiplier static errors. This same bias, though smaller in size was observed in the quaternion simulation. This leads to a strong suspicion that it is due to some characteristic of the computing equipment itself, rather than some outright mistake in the setup. To be sure, the setup was checked most carefully, many components were interchanged and all the usual checking methods were applied with no result other than to reaffirm that the equipment was correctly wired in accordance both with the circuit diagram and the equations. Then checking of the counter characteristics was started. It was thought that possibly the time base of the computer (i. e. the time constant of the integrators) was not exactly the same as that of the counter. To check this a linear 0. 5 rad/sec oscillator of 50 volt amplitude was set up using only two integrators, two hand-set potentiometers and one inverting amplifier. This oscillator was then allowed to run for 125. 66 seconds, and the result measured. The oscillator agreed precisely with the timer. The output of the oscillator integrator which had a zero initial condition (the other one had a 50 volt initial condition) went through zero within 10 milliseconds of 125. 66, and the time error we are looking for is more in the nature of 300 milliseconds. This is a surprisingly good check.

Next it was thought that it might be a phase lag effect. This was ruled out by two experiments. First, a run was made so as to allow only five complete rotations of the system rather than 10. It was found that the bias was very nearly one half that of Figure 7. If it had been a phase lag effect, it would have been more nearly constant with time. Furthermore, the phase lag of a multiplier was measured directly at this frequency, and was found to be about 0. 6 milliradians. The phase shift required to explain this bias would be more in the order of 0. 1 radians.

By setting up a separate oscillator using two integrators and two servos, the difficulty was finally traced to the fact that the summing and inverting amplifier gains are consistently low. The gains were carefully adjusted in this separate oscillator, and most of the bias was removed. It was not considered worthwhile to similarly trim all the summers and inverters in the entire simulation.

After all, the aim of this program was to show typical simulator results. Still, if the ultimate accuracy of which the servos are capable is to be attained, something of this nature should be done. It should be mentioned that not all of the amplifiers would have to be trimmed. Only those amplifiers which are between the P, Q, and R multiplications and integrator inputs are critical. None of those in the correction loops can give trouble. This is true of both methods. It is rather surprising to find that servos, generally viewed with suspicion and avoided when possible, should not prove to be the major source of error in these simulations. Amplifiers, rather, have proved to be the limiting factor.

It may also be shown from the data of Table 2, that the zero-input drift of this method, at least on the day the runs were made, was about 3×10^{-3} deg/sec. This is comparable with the better set of results obtained with quaternions.

SECTION VIII

SUMMARY AND CONCLUSIONS

Having investigated both quaternion and direction cosine coordinate conversions, it is now in order to compare the strengths and weaknesses of the two methods. For comparison purposes, the Euler angle method will be included, though, as mentioned earlier, it is not strictly comparable with the other two in capability. Before proceeding with this comparison, it is well to outline the criteria. It seems that there are two main areas of comparison. Listed in order of their importance they are (1) minimum equipment requirement (both qualitative and quantitative) and (2) ease of programming, reliability and comprehensibility.

As far as qualitative requirements on equipment is concerned, it has been shown that both direction cosine and quaternion methods are equally sensitive to multiplier errors. It further seems clear that since the Euler angle system (as proposed by Howe and Gilbert)* makes use of the same type of computing oscillator, it will be about the same as the other two in this respect so long as the orientation is well away from the singular point. The dependence on error increases as the singular point is approached, however, until multiplier errors dominate the solution.

The fact that the multipliers oscillate only half as fast in the quaternion method is of the first importance. There is no limit to the speed at which the servos may be required to run in the Euler angle method, though Howe and Gilbert* show surprisingly good results when servos are slewing.

It should be observed that the bandpass of even the rather old servos used in this simulation is so good that it contributes nothing to the errors, so long as the servos are operating linearly. The real problem in servos is rate and acceleration limiting. It is on this point, then, that the lower servo frequency of the quaternion method shows its real advantage.

---

* Howe and Gilbert, op. cit.

As to the amount of computing equipment required, the table below will serve as a useful means of comparison.

Method	Basic Equipment			First Vector	Each Additional Vector
	Amplifiers	Integrators	Multiplications		
Quaternion	36	4	26	9 indep	9 dep
Dir. Cosine	31	6	36	9 dep	9 dep
Euler Angle	16	6	24	12 dep	12 dep

The figures for Euler angles were taken from Howe and Gilbert with addition of another loop to compute sin ψ and cos ψ, which would be necessary if complete vector transformations were to be made. It may be seen that in amount of equipment, the advantage lies with Euler angles, with the quaternion method second, well ahead of direction cosines. The last two colums give the number and type of additional multiplications required to transfer the first vector, and each additional vector. In the all-important area of multiplications, it is seen that the quaternion method is nearly as good as direction cosines and will be better if a large number of vectors are to be transformed.

In ease of programming and reliability there is not much to choose between quaternions and direction cosines, except that the latter takes somewhat more equipment. In the Euler angle system, some thought must be given to keeping the inevitable division circuit stable and possibly protecting the associated amplifiers, but this is not serious. The high card of Euler angles is that they are so easy to interpret. After all, a prime function of a simulator is to tell the operator or engineer what the simulated system is doing, and for ready interpretation, there is nothing like the Euler angle system. In the first place, most people in the aircraft field know what "pitch", "roll", and "bank" mean whether they know differential equations or not. Cockpit presentations of attitudes are given in terms of gimbal angles, which are nothing but Euler angles. There is a tendency to overrate the advantage, however. It is a matter of experience, and one can learn to interpret a transformation matrix with only a [illegible] of effort. Similarly one can learn to interpret the

Finally, then, a simple statement of the comparison is this: Euler angles are convenient for interpretation, but for accuracy, they cannot compete with either direction cosines or quaternions. On all significant bases of comparison, the quaternion method appears superior to direction cosines. Of course, it should be kept in mind that we are considering only coordinate transformations capable of unrestricted rotation about any axis.

It might not be out of place at this point to interject a few remarks about the general use of three-dimensional coordinate conversions of the type considered here. It goes almost without saying, that such a coordinate conversion would not be used except in a very large and involved simulation. Another way of putting it is that in order to make this type of coordinate conversion useful, other parts of the problem must receive a similarly general and unrestricted treatment. This is very rarely done. The reason is that the amount of labor involved is such as to be justifiable by only the most overwhelming technical reasons. Computer capacity is not the limiting factor. There are many computer installations in the country whose capacity is equal to the largest simulations yet attempted. The problem appears to lie in the tremendous amount of painstaking detail involved to set up the problem, check it out and keep it working. The only reasonable answer to this is a policy of programming problems in larger pieces. This is comparable to use of subroutines on a digital machine. Without going into the matter at length, it seems clear that coordinate conversion lends itself to this technique probably better than any other part of the problem. Consider for example, the quaternion method of Figure 2. The inputs to the coordinate conversion are the three voltages P, Q, and R, and the outputs are the nine direction cosines. There are only twelve gains which would have to be changed from one problem to the next, and these are the P, Q, and R product inputs to the four integrators. These serve to establish the maximum allowable rate of rotation and the scale factor on the inputs P, Q, and R. Nothing else in the entire circuit would have to be changed. The computer operator would not even have to know how the coordinate conversion worked. It would be converted into that "black box" of which we are all so fond. By using this sort of technique throughout the problem, the amount of labor involved in large simulations could be cut by the factor ten. This would simplify things other than the setup procedure. The engineer would no longer have to spend so much time deciding what is negligible. He could start everything in and find what is negligible by throwing it out and seeing if it changes the result, which, of course is in the best mathematical tradition.

APPENDIX A

ORTHOGONAL TRANSFORMATIONS

1 The Independent Coordinates of a Rigid Body

Fundamental to the study of rigid body motions is the determination of how many degrees of freedom it has. Putting it another way, the problem is to determine how many numbers one must specify in order to describe the orientation of the body. In order to do this, it will also be necessary to give a more exact definition to the term "rigid".

Assume that a body is composed of a large number of elementary particles. If the distance between the i th particle and the j th particle r_{ij} is constant of all particles i and j, then the body is said to be rigid. If all the N particles were independent of each other, it would require 3N coordinates to specify them all. (Three cartesian coordinates are required to specify the position of a point.) The particles are not all independent, however. In fact the position of any particle in the body may be specified by the distances to any three non-collinear points in the body.

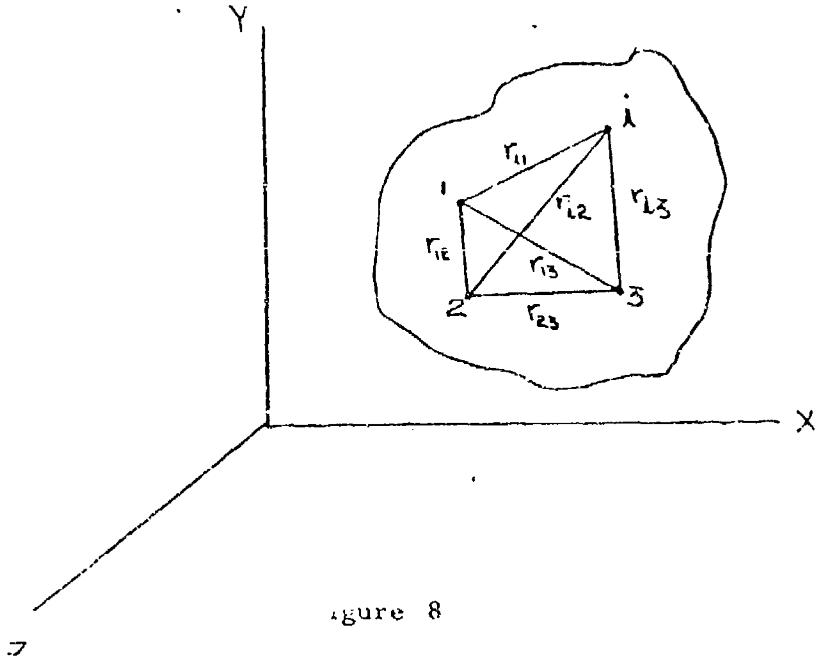


Figure 8

The points 1, 2 and 3 in Figure 8 have been chosen at random, the only condition being that they do not lie along the same line. By the rigid body condition that r_{i1}, r_{i2} and r_{i3} are constant, the position of the i th particle is fixed once the positions of the particles i, it follows that the position of every particle in the body is specified once the three points are specified. In other words, the position of the body is specified by the positions of these three points. Specifying three points would require nine coordinates if all the points were independent.

There are three conditions to be fulfilled by these coordinates, however, namely the prescribed values of r_{12}, r_{13} and r_{23}. Thus six coordinates are required to specify the position of the rigid body. Another way of saying this is to say that the rigid body has six degrees of freedom. These are frequently divided into two groups called translational and rotational degrees of freedom. The three coordinates used to specify the orientation of some point in the body (say the point 1 in Figure 8) in the xyz coordinate system, may be called the translational coordinates, while the three coordinates required to specify the relative orientation of the other two points could be called the rotational coordinates. The translational coordinates, then, are associated with the motion of the body as a whole, while the rotational coordinates are associated with the orientation of the body.

2. Orthogonal Transformations

Consider a vector $\vec{r}$ which has components x, y and z in the XYZ coordinate system. If the unit vectors along the X, Y and Z axes are called $\vec{i}$, $\vec{j}$ and $\vec{k}$, then it is possible to write r as follows:

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z \tag{122}$$

Now assume some coordinate system X'Y'Z' which has the same origin as the XYZ system but an arbitrary rotation with respect to it. The components of $\vec{r}$ in this system are x' y' and z' and the unit vectors along the three axes are $\vec{i}'$, $\vec{j}'$ and $\vec{k}'$. The vector $\vec{r}$ may also be written

$$\vec{r} = \vec{i}'x' + \vec{j}'y' + \vec{k}'z'. \tag{123}$$

The problem is to determine the components x', y' and z' in terms of x, y and z and the relative orientation of the two coordinate systems. This process is called an orthogonal transformation.

It is possible to write the unit vector $\vec{i'}$ in terms of its components in the XYZ system,

$$\vec{i'} = (\vec{i'}\cdot\vec{i})\vec{i} + (\vec{i'}\cdot\vec{j})\vec{j} + (\vec{i'}\cdot\vec{k})\vec{k}. \tag{124}$$

Since all these vectors have unit magnitude, the dot product of two is simply the cosine of the angle between them.

$$\begin{aligned} \vec{i'}\cdot\vec{i} &= \cos \angle \vec{i'}\cdot\vec{i} = a_{11}, \\ \vec{i'}\cdot\vec{j} &= \cos \angle \vec{i'}\cdot\vec{j} = a_{12}, \\ \vec{i'}\cdot\vec{k} &= \cos \angle \vec{i'}\cdot\vec{k} = a_{13}. \end{aligned} \tag{125}$$

The same process may be applied in obtaining $\vec{j'}$ and $\vec{k'}$.

$$\vec{j'} = (\vec{j'}\cdot\vec{i})\vec{i} + (\vec{j'} + \vec{j})\vec{j} + (\vec{j'}\cdot\vec{k})\vec{k},$$

$$\vec{k'} = (\vec{k'}\cdot\vec{i})\vec{i} + (\vec{k'}\cdot\vec{j})\vec{j} + (\vec{k'}\cdot\vec{k})\vec{k},$$

so the entire set of relationships may be written:

$$\begin{aligned} \vec{i'} &= a_{11}\vec{i} + a_{12}\vec{j} + a_{13}\vec{k}, \\ \vec{j'} &= a_{21}\vec{i} + a_{22}\vec{j} + a_{23}\vec{k}, \\ \vec{k'} &= a_{31}\vec{i} + a_{32}\vec{j} + a_{33}\vec{k}. \end{aligned} \tag{126}$$

It is possible to apply an exactly similar process in expressing the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ in terms of their components in the X'Y'Z' system.

$$\begin{aligned} \vec{i} &= a_{11}\vec{i'} + a_{21}\vec{j'} + a_{31}\vec{k'}, \\ \vec{j} &= a_{12}\vec{i'} + a_{22}\vec{j'} + a_{32}\vec{k'}, \\ \vec{k} &= a_{13}\vec{i'} + a_{23}\vec{j'} + a_{33}\vec{k'}. \end{aligned} \tag{127}$$

Figure 9 shows the two coordinate systems and the unit vectors.

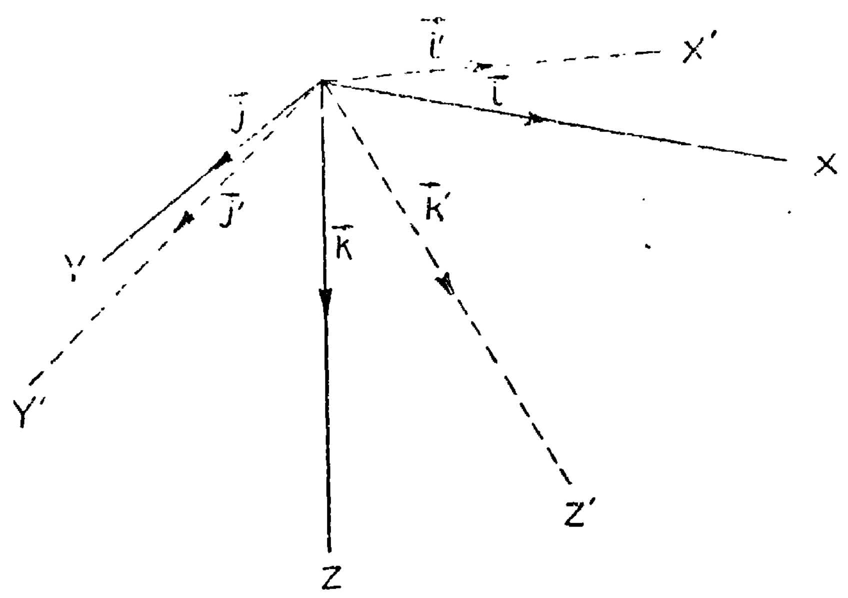


Figure 9.

It is now possible to determine the components of the vector $\vec{r}$ in the X'Y'Z' coordinate system.

$$
\begin{aligned}
x' &= \vec{r} \cdot \vec{i}' = a_{11}x + a_{12}y + a_{13}z, \\
y' &= \vec{r} \cdot \vec{j}' = a_{21}x + a_{22}y + a_{23}z, \\
z' &= \vec{r} \cdot \vec{k}' = a_{31}x + a_{32}y + a_{33}z.
\end{aligned} \tag{128}
$$

The nine quantities a_{11} - - a_{33} are called the direction cosines. They provide the means of transforming a vector from one coordinate system to another and therefore they specify the orientation of the X'Y'Z' system with respect to the XYZ system. It was developed earlier that only three parameters were necessary to specify the orientation of a rigid body. Therefore there must be six equations relating the direction cosines to each other. It will be noted that regardless of what rotation is applied to the coordinate system, the length of any vector must remain unchanged. This means that

$$(x')^2 + (y')^2 + (z')^2 = x^2 + y^2 + z^2. \tag{129}$$

Substitution of the Equations (128) into this equation shows that if Equation (129) is to hold identically for all values of x, y and z, then the following conditions must obtain:

$$
\begin{aligned}
a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1 , \\
a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1 , \\
a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1 , \\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0 , \\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0 , \\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0 .
\end{aligned}
\qquad (130)
$$

These six equations are called the orthogonality conditions. The entire set of equations may be written in condensed form as

$$\sum_i a_{ij}a_{ik} = \delta_{jk} , \qquad (131)$$

where δ_{jk} is the Kronecker δ-symbol which is defined by

$$
\begin{aligned}
\delta_{jk} &= 1 , \quad (j = k) \\
\delta_{jk} &= 0 , \quad (j \neq k).
\end{aligned}
\qquad (132)
$$

It will be noted that the nine direction cosines, restrained by the six orthogonality equations give the three independent parameters necessary to define the orientation of a rigid body. The nine direction cosines may be written in an array called a matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{pmatrix} = (A). \qquad (133)$$

Matrices are a type of mathematical entity which may be conveniently applied to the problem of rigid body rotations. The rules for manipulating these quantities will now be reviewed.

3. Properties of Matrices

The multiplication of a matrix by a vector is the first operation of interest. Symbolically, this is represented by

$$\vec{r}' = (A)\vec{r}. \tag{134}$$

For convenience, the x, y and z components of $\vec{r}$ are denoted by x_1 x_2 and x_3. Note that a vector $\vec{r}$ may be viewed as a matrix of only one column. The equation might be written

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{135}$$

The rule for performing this operation is

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j . \tag{136}$$

If these operations are carried out, a set of three equations is obtained which is identical with the set of Equations (128). This means that multiplication of a vector by matrix using the multiplication rule above represents a transformation of that vector from one coordinate system to another. For this reason, the matrix (A) may be called the transformation matrix.

The case of two successive rotations is an important one. Let the first rotation be represented by a matrix (B). Then the components of a vector after this rotation will be given by

$$x_k' = \sum_j b_{kj} x_j . \tag{137}$$

If the second rotation is represented by the matrix (A), then the components of the vector after this second rotation would be

$$x_i'' = \sum_k a_{ik} x_k' . \qquad (138)$$

Substituting (137) into (138) gives

$$x_i'' = \sum_k a_{ik} \sum_j b_{kj} x_j ,$$

$$= \sum_j \left(\sum_k a_{ik} b_{kj} \right) x_j .$$

Note that this can be put in the form of Equation 136.

$$x_i'' = \sum_j c_{ij} x_j , \qquad (139)$$

where

$$c_{ij} = \sum_k a_{ik} b_{kj} . \qquad (140)$$

Thus the two rotations may be replaced by a single rotation (C), the elements of which may be computed from (140). Symbolically,

$$(C) = (A)\ (B) . \qquad (141)$$

It can be seen by the rule of Equation (140) that

$$(A)\,(B) \neq (B)\,(A) ,$$

so the process of matrix multiplication is not commutative. The process of matrix multiplication is associative.

$$(A)\ [(B)\,(C)] = [(A)\,(B)]\ (C).$$

The matrix (A) was used to transform the vector $\vec{r}$ into the vector $\vec{r}'$.

It is of interest now to investigate the properties of the matrix $(A)^{-1}$ which transforms $\vec{r}'$ into $\vec{r}$. The elements of this inverse matrix are desig- [illegible] a [illegible]. The inverse matrix is defined by the following equation.

$$(A)^{-1}\ (A)\vec{r} = \vec{r}. \qquad (142)$$

Doing the first operation, the result is

$$x_i' = \sum_j a_{ij} x_j \, . \tag{143}$$

Now applying the inverse transformation to this gives

$$x_k'' = \sum_i a_{ki}' \, x_i' \, ,$$

$$= \sum_i a_{ki}' \sum_j a_{ij} \, x_j \, ,$$

$$x_k'' = \sum_j \left(\sum_i a_{ki}' \, a_{ij} \right) x_j \, .$$

Now according to the requirement that this must give back the original vector, $x_k'' = x_k$. This will be true only if

$$\sum_i a_{ki}' \, a_{ij} = \delta_{jk} \, . \tag{144}$$

This shows that the product of the two matrices (A) and $(A)^{-1}$ will be

$$(A)^{-1} (A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (I). \tag{145}$$

This matrix (I) is called the identity matrix. It may be easily shown from the rules of matrix multiplication that for any matrix (Q),

$$(I) \; (Q) = (Q) \; (I) = (Q) \tag{146}$$

Now since $(A)^{-1}$ corresponds to some physical rotation; there must exist some matrix (R) which is the inverse of $(A)^{-1}$. In other words, there must be an (R) such that

$$(R) \; (A)^{-1} = (I). \tag{147}$$

Now if (R) is applied to both sides of Equation (145), the result is

$$(R) \; (A)^{-1} (A) = (R) \, (I) = (R). \tag{148}$$

Since matrix multiplication is associative, Equation (147) may be substituted into Equation (148) to give

$$(I)\ (A) = (R)\ ,$$
$$(A) = (R)\ .$$

This means that

$$(A)^{-1}(A) = (A)(A)^{-1} = (I)\ , \qquad (149)$$

so that (A) and $(A)^{-1}$ commute.

Now consider the double sum,

$$\sum_{k,i} a_{ij} a_{kl}'\ a_{ki}'\ . \qquad (150)$$

This sum may be written two ways, depending on the order of summation,

$$\sum_{k} \left(\sum_{i} a_{ki}'\ a_{ij} \right) a_{kl}' = \sum_{i} \left(\sum_{k} a_{kl}'\ a_{ki}' \right) a_{ij}\ . \qquad (151)$$

Applying Equation (144) to the quantity in parentheses on the left hand side, and applying the orthogonality condition of Equation (131) to the quantity in parentheses on the right hand side, the result is

$$\sum_{k} \delta_{jk}\ a_{kl}' = \sum_{i} \delta_{li}\ a_{ij}\ , \qquad (152)$$

$$a_{jl}' = a_{lj}\ .$$

This is the important result. To form the inverse of an orthogonal matrix, the rows and columns are simply interchanged. Note that this conclusion holds true only for orthogonal matrices. This is because the orthogonality conditions were used to prove Equation (152). In general, the matrix formed by interchanging rows and columns is called the transposed matrix and is designated by $(\widetilde{A})$. The complex conjugate of this transposed matrix is called the adjoint matrix and is indicated by $(A)^{+} = (\widetilde{A})^{*}$. A matrix is said to be unitary if it satisfies the condition,

$$(A)^{+}(A) = (I)\ . \qquad (153)$$

Of course these latter definitions are relatively meaningless in the case of real matrices. However, use is sometimes made of matrices, the elements of which are complex numbers.

It is of interest to investigate the characteristics of the determinant formed by the element of a matrix. The determinant of the matrix (A) will be written as [A]. It will be noted that the law of matrix multiplication is the same as the law for multiplication of determinants. Therefore,

$$[AB] = [A]\ [B]. \tag{154}$$

Evidently the determinant of the identity matrix has the value unity, therefore, from Equation (145) it may be seen that

$$[A^{-1}]\ [A] = 1, \tag{155}$$

provided that (A) is orthogonal. Since interchanging rows and columns does not alter the value of a determinant, $[A^{-1}] = [A]$ and, from Equation (155),

$$[A]^2 = 1. \tag{156}$$

This means that the determinant of the transformation matrix can have only the values plus or minus one. If the rotation is a real one, it may be shown that +1 is the only allowable value. There is a certain type of matrix operation which is called a similarity transformation. It is defined by

$$(A)' = (B)\ (A)\ (B)^{-1}. \tag{157}$$

It can easily be shown that the determinant of (A) is the same as the determinant of (A)', that is to say, the value of the determinant of a matrix is invariant under a similarity transformation of that matrix. This may be shown by simply applying both sides of (157) to the matrix (B).

$$(A)'\ (B) = (B)\ (A)\ (B)^{-1}\ (B) = (B)\ (A). \tag{158}$$

From this it is seen that

$$[A']\ [B] = [B]\ [A]. \tag{159}$$

Since [B] is a number and not zero, it is possible to divide both sides by it and obtain the result

$$[A] = [A'], \tag{160}$$

which demonstrates the proposition.

There is another set of relationships among the direction cosines which will prove to be of interest. Consider the set of Equations (126). If the $\vec{i}'$, $\vec{j}'$ and $\vec{k}'$ vectors are mutually perpendicular, then the following relationships apply:

$$\vec{i}' \times \vec{j}' = \vec{k}', \quad \vec{j}' \times \vec{k}' = \vec{i}', \quad \vec{k}' \times \vec{i}' = \vec{j}'. \qquad (161)$$

If these vector equations are expanded in the unprimed system, and their components equated, the result is

$$\begin{aligned} a_{11} &= a_{22}a_{33}-a_{23}a_{32}, \\ a_{21} &= a_{13}a_{32}-a_{12}a_{33}, \\ a_{31} &= a_{12}a_{23}-a_{13}a_{22}, \\ a_{12} &= a_{23}a_{31}-a_{11}a_{23}, \\ a_{22} &= a_{11}a_{33}-a_{13}a_{31}, \\ a_{32} &= a_{13}a_{21}-a_{11}a_{23}, \\ a_{13} &= a_{21}a_{32}-a_{22}a_{31}, \\ a_{23} &= a_{12}a_{31}-a_{11}a_{32}, \\ a_{33} &= a_{11}a_{22}-a_{12}a_{21}. \end{aligned} \qquad (162)$$

These nine equations are really consequences of the orthogonality conditions. They present a means for solving for any direction cosine in terms of the others.

4. Infinitesmal Rotations

It would be a great advantage if a vector could be associated with a finite rotation, but it turns out that this is not possible. For one thing, finite rotations are not commutative, nor even anti-commutative. That is to say the order of the operations must be preserved. While this is true of a finite rotation, it will be shown that a vector may be associated with an infinitesmal rotation and that therefore, the known characteristics of vectors may be used in the treatment of such rotations. Consider the matrix that describes a rotation thru the angle $\Delta\mu$ [illegible] which makes the angles α_1, β_1 and γ_1 with the X, Y and Z axes respectively. This matrix may be gotten by substituting

$$\sin \frac{\Delta\mu_1}{2} = \frac{\Delta\mu_1}{2}, \quad \cos \frac{\Delta\mu_1}{2} = 0$$

into the matrix (12) and dropping higher order terms. The result is

$$(A)_1 = \begin{pmatrix} 1 & \Delta\mu_1 \cos \gamma_1 & -\Delta\mu_1 \cos \beta_1 \\ -\Delta\mu_1 \cos \gamma_1 & 1 & \Delta\mu_1 \cos a_1 \\ \Delta\mu_1 \cos \beta_1 & -\Delta\mu_1 \cos a_1 & 1 \end{pmatrix}. \qquad (163)$$

This matrix differs only slightly from the identity matrix. This may be seen more clearly by writing it in the following form:

$$(A)_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \Delta\mu_1 \cos \gamma_1 & -\Delta\mu_1 \cos \beta_1 \\ -\Delta\mu_1 \cos \gamma_1 & 0 & \Delta\mu_1 \cos a_1 \\ \Delta\mu_1 \cos \beta_1 & -\Delta\mu_1 \cos a_1 & 0 \end{pmatrix}. \qquad (164)$$

This latter matrix is anti-symmetric or skew-symmetric. Notice that this matrix has only three independent elements, $\Delta\mu_1 \cos a_1$; $\Delta\mu_1 \cos \beta_1$; $\Delta\mu_1 \cos \gamma_1$ and that they are simply the three components of a vector of magnitude $\Delta\mu$ which is oriented along the axis of rotation. It will be shown that this is the vector which may be associated with infintesmal rotation. Let these three components be called Ω_1, Ω_2, Ω_3 so that (101) may be written

$$(A)_1 = (I) + \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & -\Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}. \qquad (165)$$

Now if the infinitesmal rotation $(A)_1$ is followed by another infinitesmal rotation $(A)'_1$, of the form

$$(A)'_1 = (I) + \begin{pmatrix} 0 & \Omega'_3 & -\Omega'_2 \\ -\Omega'_3 & 0 & \Omega'_1 \\ \Omega'_2 & -\Omega'_1 & 0 \end{pmatrix}, \qquad (166)$$

then the combined rotation $(A)_1'(A)_1$ is seen to be the following, if higher order infinitesmals are dropped;

$$(A)_1'' = (A)_1'\ (A) = (I) + \begin{pmatrix} 0 & \Omega_3'' & -\Omega_2'' \\ -\Omega_3'' & 0 & \Omega_1'' \\ \Omega_2'' & -\Omega_1'' & 0 \end{pmatrix}, \qquad (167)$$

where

$$\Omega_3'' = \Omega_3' + \Omega_3\ ; \quad \Omega_2'' = \Omega_2' + \Omega_2\ ; \quad \Omega_1'' = \Omega_1' + \Omega_1\ .$$

Since the second order infinitesmals were dropped, the order or sequence of the infinitesmal rotations is unimportant. This is one condition which is necessary if these rotations are to be represented by vectors. From the makeup of Ω_1'', Ω_2'', Ω_3'', it is seen that the vector representing the combined rotation is simply the sum of the two vectors for the single rotations.

A more conclusive demonstration of the fact that the quantities Ω_1, Ω_2, Ω_3 are the components of a vector associated with the infinitesmal transformation is the demonstration that the matrix components transform like components of a vector under a coordinate transformation. Consider a matrix (A) which operates on a vector $\vec{R}$ to produce a vector $\vec{R}'$.

$$\vec{R}' = (A)\vec{R}. \qquad (168)$$

Now if an additional matrix (B) is applied to this equation,

$$\begin{aligned} (B)\vec{R}' &= (B)\,(A)\vec{R}, \\ (B)\vec{R}' &= (B)\,(A)\,(B)^{-1}\,(B)\vec{R}. \end{aligned} \qquad (169)$$

This equation is simply Equation (168) when seen in a different coordinate system, and $(B)\,(A)\,(B)^{-1}$ is the matrix (A) when viewed from the different coordinate system. This is the similarity transformation, which has been introduced before. If a similarity transformation is applied to the 'matrix of Equation (165), the result is

$$(B)\,(A)_1\,(B)^{-1} = (A)_1' = (I) + \begin{pmatrix} 0 & \Omega_3' & -\Omega_2' \\ -\Omega_3' & 0 & \Omega_1' \\ \Omega_2' & -\Omega_1' & 0 \end{pmatrix}. \qquad (170)$$

Expanding and equating components,

$$
\begin{aligned}
\Omega_1' &= b_{11}\,\Omega_1 + b_{12}\,\Omega_2 + b_{13}\,\Omega_3,\\
\Omega_2' &= b_{21}\,\Omega_1 + b_{22}\,\Omega_2 + b_{23}\,\Omega_3,\\
\Omega_3' &= b_{31}\,\Omega_1 + b_{32}\,\Omega_2 + b_{33}\,\Omega_3.
\end{aligned}
\qquad (171)
$$

Thus, the infinitesmal transformation, when viewed from the other coordinate system defined by (B) is still nearly the identity transformation, and the vector which represents the vector associated with the infinesmal transformation in this new system is simply the transform of the vector representing the infinitesmal transformation in the other coordinate system. This shows the vector character of the set of elements Ω_1, Ω_2, Ω_3.

By using this infinitesmal transformation, the rate of change of the transformation matrix (A) may be found in much the same way that the derivative of the matrix (H) was established in Section III. If (A) is the matrix at the beginning of time interval, and (A) is the matrix at the end of time Δt, then the derivative of (A) is given by

$$
\frac{d}{dt}(A) = \lim_{\Delta \to 0}\left\{\frac{(A)' - (A)}{\Delta t}\right\}. \qquad (172)
$$

(A)' may be viewed as the rotation (A) followed by the infinitesmal transformation going from (A) to (A)'. In other words

$$
(A)' = \left[(I) + (\epsilon)\right](A),
$$

where

$$
(\epsilon) = \Delta\mu \begin{pmatrix} 0 & \cos\gamma & -\cos\beta \\ -\cos\gamma & 0 & -\cos\alpha \\ \cos\beta & -\cos\alpha & 0 \end{pmatrix}, \qquad (173)
$$

Thus the derivative becomes

$$
\frac{d}{dt}(A) = \lim_{\Delta \to 0}\frac{(\epsilon)}{\Delta t}(A) = \lim_{\Delta \to 0}\frac{\Delta\mu}{\Delta t}\begin{pmatrix} 0 & \cos\gamma & -\cos\beta \\ -\cos\gamma & 0 & \cos\alpha \\ \cos\beta & -\cos\alpha & 0 \end{pmatrix}. \qquad (174)
$$

Again, in the limit $\frac{\Delta\mu}{\Delta t}$ is simply the rate of rotation, and $\frac{d\mu}{dt}\cos\alpha = P$, $\frac{d\mu}{dt}\cos\beta = Q$, $\frac{d\mu}{dt}\cos\gamma = R$, so the equation becomes

$$\begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} & \dot{a}_{13} \\ \dot{a}_{21} & \dot{a}_{22} & \dot{a}_{23} \\ \dot{a}_{31} & \dot{a}_{32} & \dot{a}_{33} \end{pmatrix} = \begin{pmatrix} 0 & R & -Q \\ -R & 0 & P \\ Q & -P & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} . \qquad (175)$$

Expanding, and equating components gives

$$\begin{aligned}
\dot{a}_{11} &= a_{21}R - a_{31}Q , \\
\dot{a}_{12} &= a_{22}R - a_{32}Q , \\
\dot{a}_{13} &= a_{23}R - a_{33}Q , \\
\dot{a}_{21} &= a_{31}P - a_{11}R , \\
\dot{a}_{22} &= a_{32}P - a_{12}R , \\
\dot{a}_{23} &= a_{33}P - a_{13}R , \\
\dot{a}_{31} &= a_{11}Q - a_{21}P , \\
\dot{a}_{32} &= a_{12}Q - a_{22}P , \\
\dot{a}_{33} &= a_{13}A - a_{23}P .
\end{aligned} \qquad (176)$$

These are the rates of change of the direction cosines in terms of the angular velocity. Now if Equation (175) be multiplied on the right by the transpose of (A), the result is

$$\begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} & \dot{a}_{13} \\ \dot{a}_{21} & \dot{a}_{22} & \dot{a}_{23} \\ \dot{a}_{31} & \dot{a}_{32} & \dot{a}_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & R & -Q \\ -R & 0 & P \\ Q & -P & 0 \end{pmatrix} . \qquad (177)$$

Expanding and equating components gives the following relationships:

$$\begin{aligned}
P &= a_{31}\dot{a}_{21} + a_{32}\dot{a}_{22} + a_{33}\dot{a}_{23} , \\
-P &= a_{21}\dot{a}_{31} + a_{22}\dot{a}_{32} + a_{23}\dot{a}_{33} , \\
Q &= a_{11}\dot{a}_{31} + a_{12}\dot{a}_{32} + a_{13}\dot{a}_{33} , \\
-Q &= a_{31}\dot{a}_{11} + a_{32}\dot{a}_{12} + a_{33}\dot{a}_{13} , \\
R &= a_{21}\dot{a}_{11} + a_{22}\dot{a}_{12} + a_{23}\dot{a}_{13} , \\
-R &= a_{11}\dot{a}_{21} + a_{12}\dot{a}_{22} + a_{13}\dot{a}_{23} .
\end{aligned} \qquad (178)$$

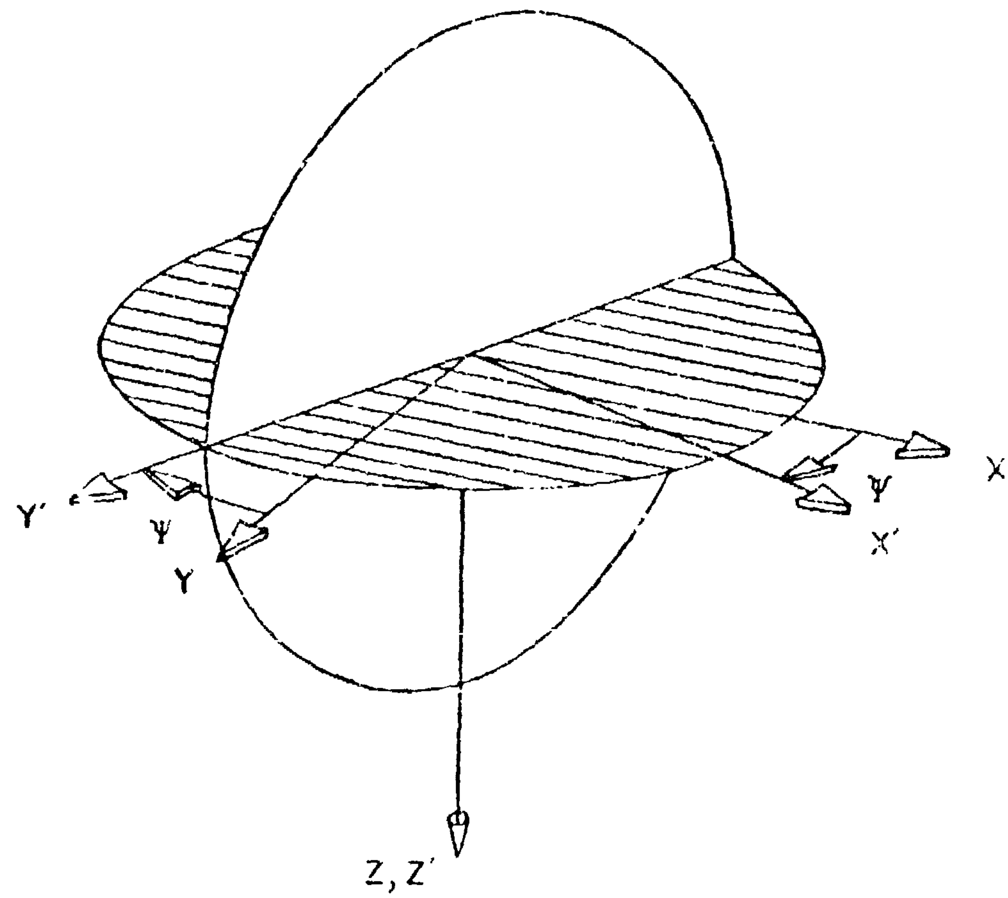
It is interesting that two different expressions are obtained for each of the velocity components. This is a consequence of the great amount of redundancy in the direction cosines. The equivalence of the two expressions for any one of the components may be shown by making use of Equation (162).

APPENDIX B

THE EULER ANGLES

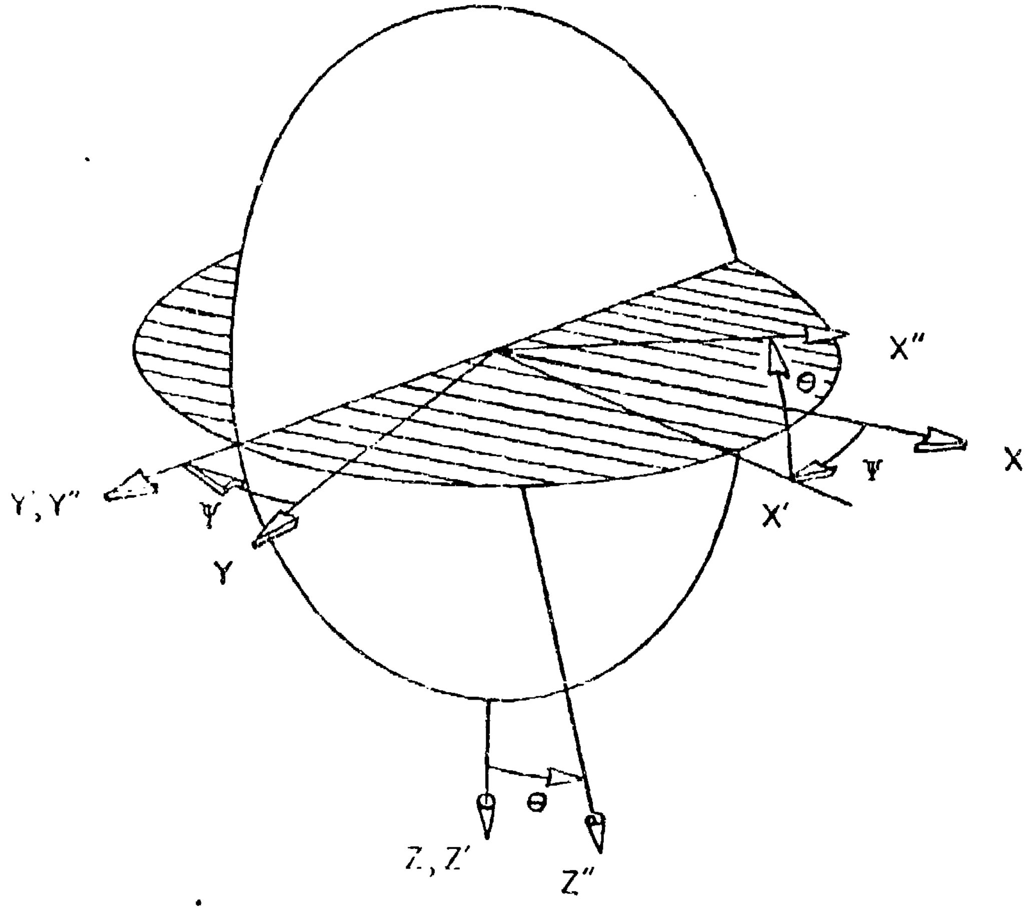
It was demonstrated in Appendix A that three parameters were required to fix the orientation of a rigid body and hence of a coordinate system. The nine direction cosines do not lend themselves to a reduction to three simple parameters, nor do they give a very lively picture of the orientation of the body. Both these difficulties are overcome by use of Euler angles, the only three-parameter system in common use. In this method, a rotation is represented by three individual rotations taken in a specified sequence about certain specific axes. In the literature, there is no agreement whatever on the order of rotations, the axes about which the rotations are made, or notation. These are varied to suit the needs of the problem and/or the author's whim. Texts on classical mechanics give sets of angles defined so as to facilitate solution of the spinning top problem. The system presented here is the most common, though by no means the only one used in aircraft work.

Consider two coordinate systems initially coincident. One set of coordinates, the x, y, z, will be referred to as the fixed system, and the other will move with respect to it. The first rotation is through the angle ψ about the z' axis. This is shown in Figure 10.



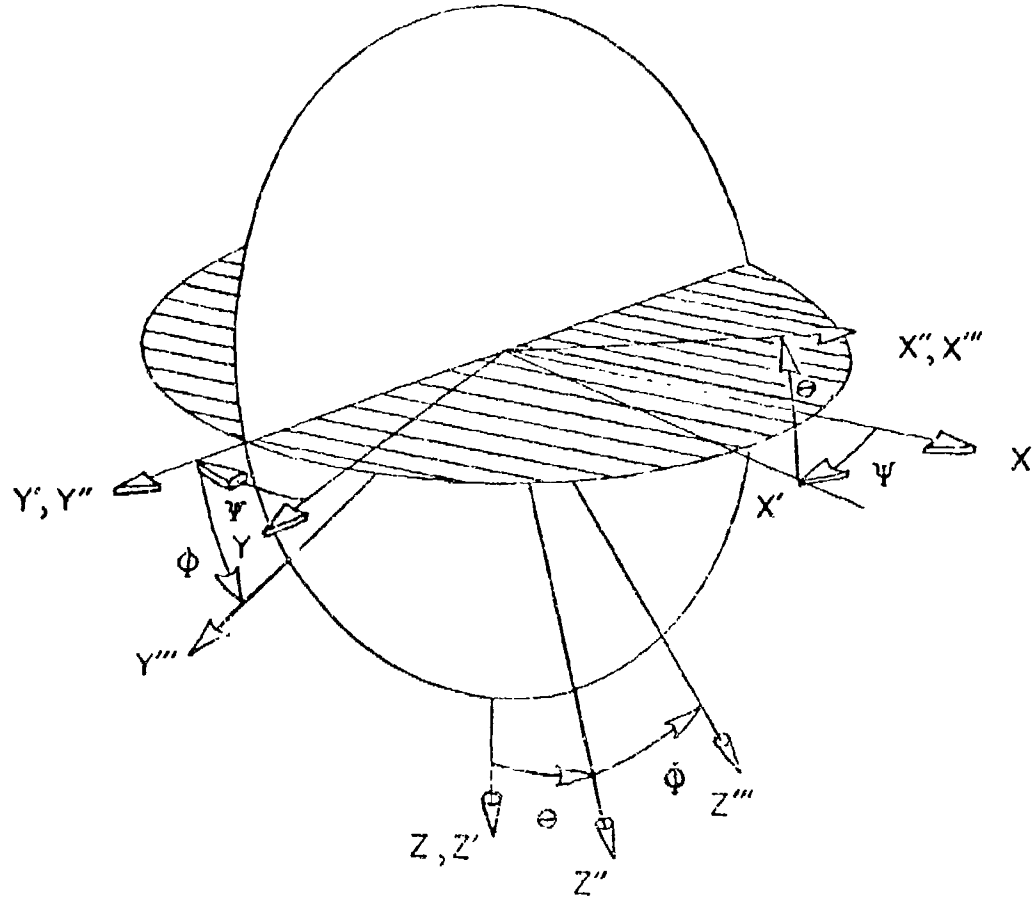
FIRST ROTATION - YAW ANGLE Ψ

Figure 10



SECOND ROTATION-PITCH ANGLE θ

Figure 11



THIRD ROTATION - ROLL ANGLE Φ

Figure 12

The second rotation is through the angle θ and is done about the Y' axis and the resulting axis system is called X'', Y'', Z''. This rotation is shown in Figure 11. θ is commonly called the pitch angle. The final rotation is done about the X'' axis through the angle ϕ. This is called the roll angle and all three rotations are shown in Figure 12. Note that all three of these rotations are in the positive sense. That is to say if the thumb of the right hand is placed along the axis of rotation, the direction of rotation is that direction in which the curled fingers point.

It is now necessary to determine the transformation matrix in terms of these Euler angles. It was shown earlier that successive rotations could be represented by a matrix which is a product of the matrices of the individual rotations. It is necessary then, only to compute the matrix corresponding to each of the Euler angle rotations and to multiply them together in the appropriate order. Note that each of the rotations is simply a two-dimensional transformation because in each case the rotation is about one of the moving axes and hence components along that axis are unchanged.

Consider first, the rotation through the angle which is shown in Figure 10. If this is viewed from above, the transformation of some arbitrary vector R would appear as shown in Figure 13

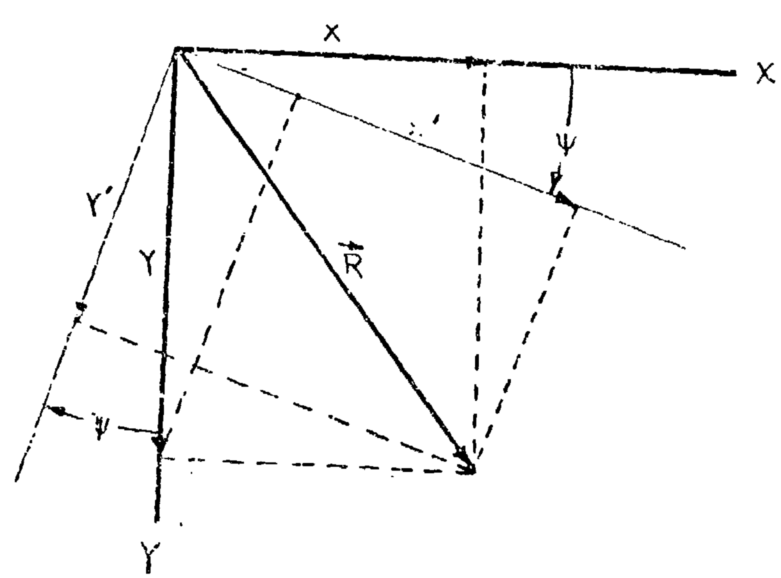


Figure 13

It can be seen from the geometry of Figure 13 that the new x' and y' components are related to the old by the equations

$$
\begin{aligned}
x' &= x \cos \psi + y \sin \psi , \\
y' &= -x \sin \psi + y \cos \psi .
\end{aligned}
\qquad (179)
$$

Since the rotation was about the z axis, any z component of R would remain unchanged. In other words, Z = Z'. This fact, together with the Equation (179) shows that the matrix for the rotation is

$$
\begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (180)
$$

Now the rotation of Figure 11 may be viewed from the front along the Y' axis, and Figure 14 is obtained.

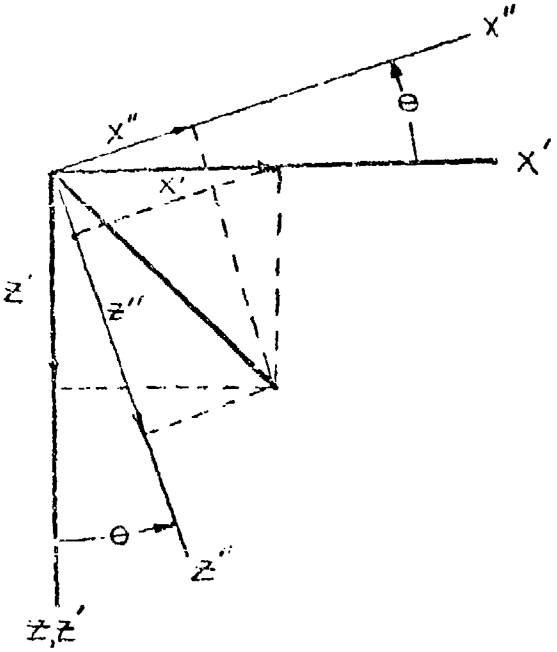


Figure 14

From the geometry of the above figure, it may be seen that

$$
\begin{aligned}
x'' &= -x' \cos \theta - z' \sin \theta , \\
z'' &= -x' \sin \theta + z' \cos \theta .
\end{aligned}
\qquad (181)
$$

In this rotation, the Y components remain unchanged so that Y'' = Y'. Therefore, the matrix for this rotation is

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} . \tag{182}$$

The final rotation may be viewed from the front, looking along the X'' axis of Figure 12.

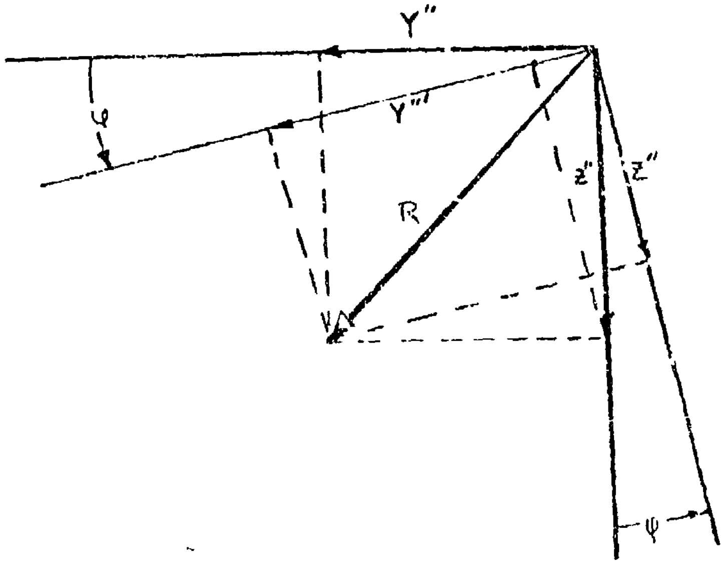


Figure 15

From the geometry of this figure it is seen that

$$\begin{aligned} y''' &= y'' \cos\phi + z'' \sin\phi, \\ z''' &= -y'' \sin\phi + z'' \cos\phi. \end{aligned} \tag{183}$$

In this rotation, the X components remain unchanged so that X''' = X''. Therefore, the transformation matrix for this rotation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} . \tag{184}$$

In order to get the total transformation matrix which results from these three rotations, it is only necessary to multiply the three individual matrices in the correct order.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi\cos\phi + \sin\phi\sin\theta\cos\psi & \cos\psi\cos\phi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ +\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\theta\cos\phi \end{pmatrix} . \qquad (185)$$

By comparison of this matrix with the matrix (A) it may be seen that all of the direction cosines and hence the complete transformation, can be expressed in terms of the three independent parameters ψ, θ, ϕ.

Since the position of a coordinate system may be specified in terms of Euler angles, the rate of rotation of that coordinate system must be related to the rates of change of the Euler angles. We now investigate this relationship.

It is shown in Appendix A that a vector could be associated with a rate of rotation. This vector is along the instantaneous axis of rotation and is equal in magnitude to the rate of rotation. Thus, each of the Euler angle rates may be associated with a vector along the axis of rotation. Observe that the vector associated with the $\dot{\psi}$ rotation of Figure 10 is directed along the Z axis and points downward if $\dot{\psi}$ is positive. Similarly, the rate of rotation due to the θ rotation of Figure 11 is a vector along the Y' axis, and if θ is increasing, the vector is in the positive y' direction. Finally, a positive roll rotation is a vector directed along the X''' axis of Figure 12. The three vectors representing the three individual Euler angles rates must be added together in order to get the entire rate of rotation of the system. Recall that these vectors are added according to the usual vector rule. The situation is shown in Figure 16 where all the Euler angle rates are assumed positive. Note that these three vectors are not mutually orthogonal. The $\dot{\psi}$ vector is normal to the $\dot{\theta}$ vector, and the $\dot{\theta}$ vector is normal to the $\dot{\phi}$ vector, but the $\dot{\phi}$ vector is not normal to the $\dot{\psi}$ vector. In any case, the three may be transformed into the X''' Y''' Z''' and added to give the entire velocity vector. the $\vec{\dot{\psi}}$ vector has the components 0, 0, $\dot{\psi}$ in the XYZ system, so to transform this into the X''' Y''' Z''' system, it is necessary to apply the full transformation matrix (185) to this vector. If this is done, the result is

$$\vec{\dot{\psi}}''' = -\vec{i}''' \dot{\psi} \sin\theta + \vec{j}''' \dot{\psi} \sin\phi \cos\theta + \vec{K}''' \dot{\psi} \cos\theta \cos\phi . \tag{186}$$

Now the vector $\vec{\dot{\theta}}$ has the components 0, $\dot{\theta}$, 0 in the X'' Y'' Z'' coordinate system. In order to get this into the X''' Y''' Z''' system, it is only necessary to transform through the last of the Euler angle rotations which is defined by the matrix (184). If this is done, the result is

$$\vec{\dot{\theta}}''' = \vec{j}''' \dot{\theta} \cos\phi - \vec{K}''' \dot{\theta} \sin\phi. \tag{187}$$

The vector $\dot{\phi}$, of course, is already in the X''' Y''' Z''' system, being defined by

$$\vec{\dot{\phi}} = \vec{i}''' \dot{\phi} . \tag{188}$$

In order to get the entire velocity vector, it is only necessary to add the last three equations. If this is done, and if the total angular velocity vector is defined as $\vec{w} = \vec{i}''' P + \vec{j}''' Q + \vec{K}''' R$, then,

$$\begin{aligned} P &= \dot{\phi} - \dot{\psi} \sin\theta, \\ Q &= \dot{\theta} \cos\phi + \dot{\psi} \sin\phi \cos\theta, \\ R &= \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi. \end{aligned} \tag{189}$$

These three equations may be solved for $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ giving

$$\begin{aligned} \dot{\psi} &= R \frac{\cos\phi}{\cos\theta} + Q \frac{\sin\phi}{\cos\theta}, \\ \dot{\theta} &= Q \cos\phi - R \sin\phi, \\ \dot{\phi} &= P + Q \tan\theta \sin\phi + R \tan\theta \cos\phi . \end{aligned} \tag{190}$$

From these equations, it is easier to see the difficulties which arises, when θ approaches 90°. For this value of θ, both $\dot{\psi}$ and $\dot{\phi}$ are infinite. It is interesting to note ... ve of θ itself has no such anomalies.

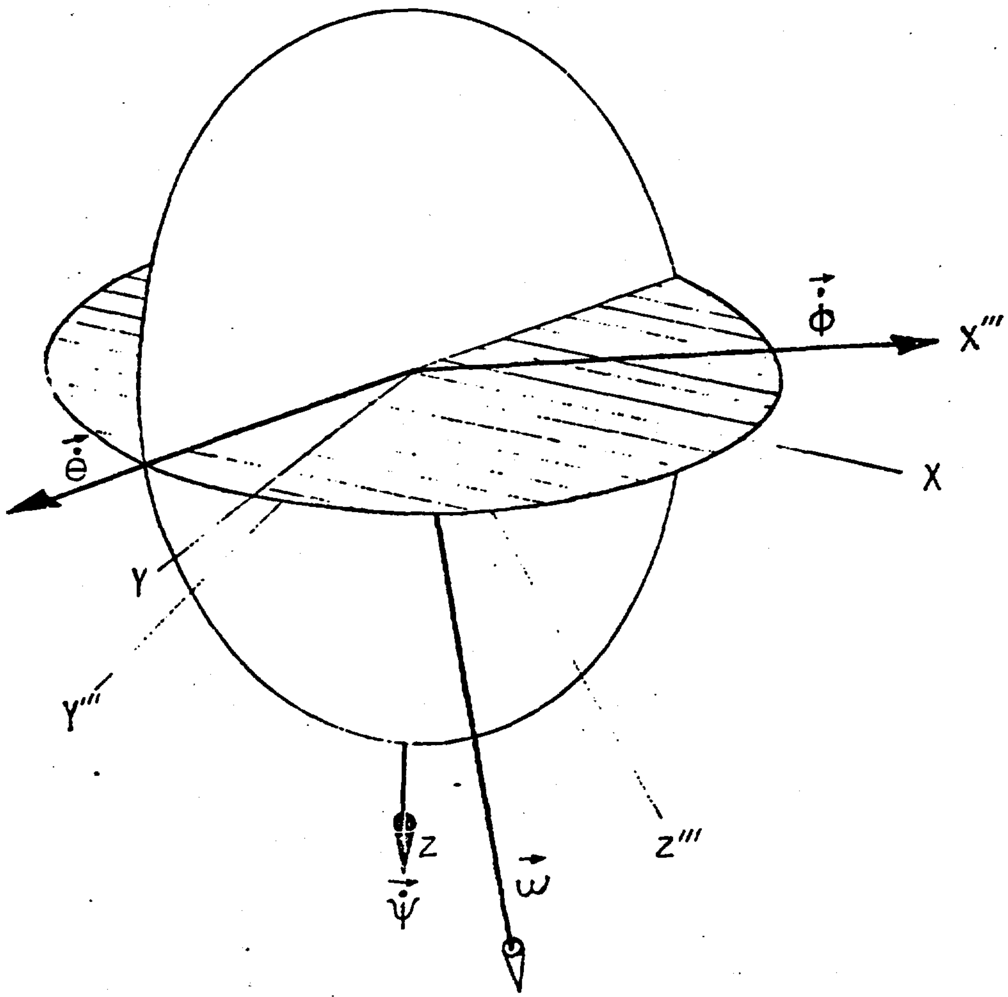


Figure 16

APPENDIX C

SERVOMULTIPLIER ERRORS

In the past, it has evidently not been the practice to treat analog multiplier errors with much care in error analyses. The usual procedure has been to assume some upper bound for the error and to consider the error constant at that value. This, of course, gives a pessimistic view of the results to be expected, though in many cases this is not undesirable, and it certainly gives an estimate of the order of magnitude of the resultant error. Possibly the principal reason that errors are not treated with more care is that little is known about their detailed character. No two multipliers are the same, a given multiplier will change with time, and it is a great bother to measure the errors anyhow. It would appear, then, that a statistical approach is indicated. If the statistical distribution of errors can be established, then it can be used to predict the distribution of errors in the problem result. This will not give a measure of the error in any particular run, but it is not practical to do this in any event. It would appear that if we know the distribution of errors in the solution, it is fair to say that we know all that is necessary about those errors.

It will become clear that the statistical procedures applied in this work are of the very simplest sort. Many more things could have been done, even with the data which were taken, but it was not felt to be worthwhile. The prinicpal reason for this is that, as was pointed out earlier, multiplier errors did not turn out to be the principal source of drift in the coordinate conversion simulation. Therefore, it was not possible to check the predicted error distribution against the observed distribution, even to the extent permitted by the small number of samples available.

Generally speaking, the procedure was to take an average of all errors of all multipliers over their entire range; taking the average in a manner generally consistent with the way the multipliers are used in the coordinate conversion simulation. It was assumed that the error voltage was directly proportional to the voltage across the multiplying potentiometer, and this was kept at ± 100 volts throughout the measurements. Once the error distribution is established for this case, it is possible to get the distribution for any other pot voltage by simply multiplying the ± 100 volt distribution by $V/100$, where V is the pot voltage.

The type of measurements made can perhaps best be illustrated by consideration of the schematic of Figure 17. This shows the circuit used to test one servo with all its pots.

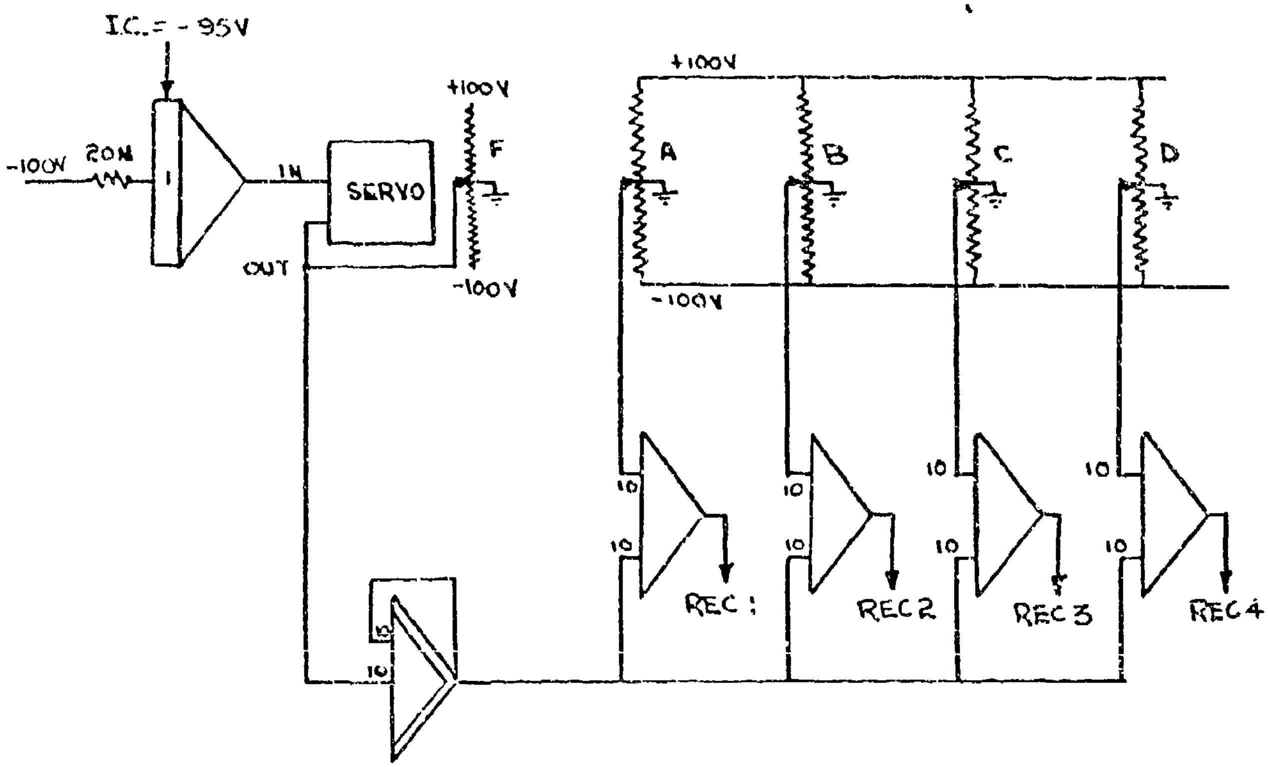


Figure 17

± 100 volts are applied across all pots. The wipers are swept linearly from one end to the other by use of the integrator. It starts at -95 volts and goes to +95 volts at the rate of approximately 5 volts/second. The entire sweep requires about 40 seconds. At this slow speed, the servo tracking error is essentially zero. It is approximately three times smaller than the pot errors. During the sweep, the voltage on each of the multiplying pots is being compared with that on the feedback pot, and the difference is recorded. All pots are loaded with 0.1 megohms. It might appear somewhat more logical to compare the multiplier pot voltage with the input rather than with the feedback pot voltage. Comparison with the input would include the tracking error. Even aside from the fact that the tracking error is small, it is actually desirable to exclude it. The point at issue here is the static accuracy of the multipliers. The tracking error is associated with the dynamic performance, which is treated by different methods. In the steady state, there will be little or no tracking error, and the only error remaining will be the difference between the feedback pot and the multiplying pot. This is exactly what is being measured by the circuit indicated.

A sample record taken in this way is shown in Figure 18. This is the one for Servo 1, and shows many features typical of the others as well. Observe that, especially on Pot A a periodicity may be seen. There are ten cycles of the oscillation, so it is evidently associated with the ten turns of the pots used in the multiplier. Not all records show this, but many do. It may also be seen that there is a rough correlation of all four traces, that is, they all tend to have the same sign at the same point, and the peaks on all traces are nearly coincident. This is probably due to the fact that, for this servo, the feedback pot is more nonlinear than the multiplier pots. On many of the records this correlation was not observed, indicating that the feedback pot was the more accurate. Also, it is seen that there are some poor contact areas on Pot A, though this was encountered in only a few cases. It should be mentioned that records of the type of Figure 18 are taken for all servos periodically by the Analog Computation Branch, Aeronautical Research Laboratory. The principal purpose of this is to check alignment of the pots, and check for noisy spots such as appear on Pot A of Figure 18. In this connection it should be pointed out again that no special adjustment of the multipliers was made for this simulation. The error records used, of which Figure 18 is an example, were not made especially for this study, but were made about two months earlier, as part of normal maintenance of the equipment. Thus the results may be considered typical of what may be obtained on this type of equipment in normal use. Especially in view of this, the multiplier accuracy does seem to be quite good.

On each of the error recordings, approximately 45 equally spaced readings were made. For each reading, the value of the error trace was sorted into class intervals, rather than being read as a number. This was done to simplify the reading process, since it was felt that all desired results could be obtained by dealing with frequencies in class intervals. All intervals were of equal width (0.02 volts). Table 3 gives the results obtained. The error voltage indicated at the top of each column is the center of the class interval.

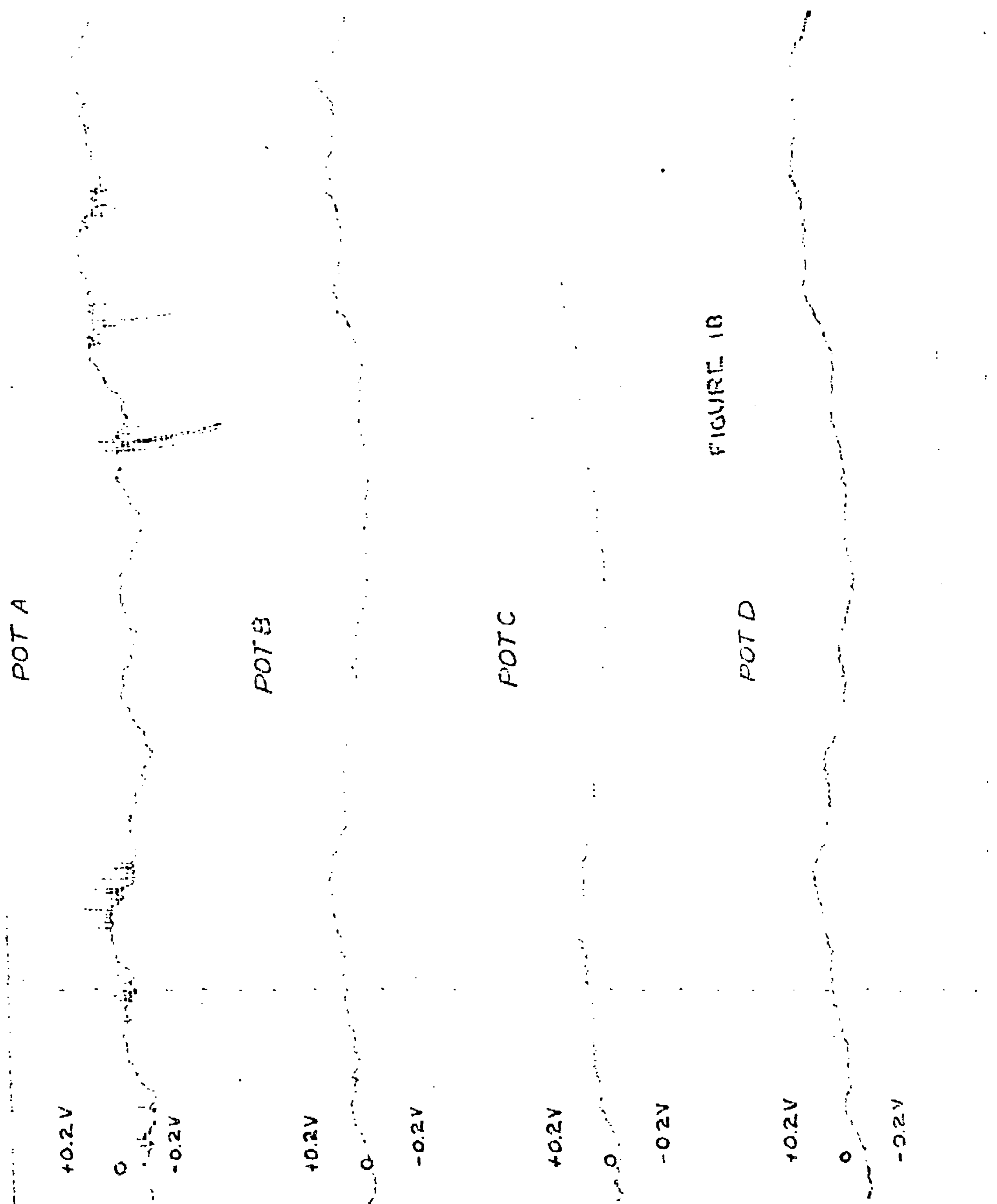


FIGURE 18
POT A
POT B
POT C
POT D
+0.2V
0
-0.2V
+0.2V
0
-0.2V
+0.2V
0
-0.2V
+0.2V
0
-0.2V

Figure 18.

TABLE 3

ERROR VOLTAGE FREQUENCIES

Servo	Pot	ERROR VOLTAGE																								No. of Observations
		-.18	-.16	-.14	-.12	-.1	-.08	-.06	-.04	-.02	0	+.02	+.04	+.06	+.08	+.1	+.12	+.14	+.16	+.18	+.2	+.22	+.24	+.26	+.28	+.3	
1	A					1	2	2	2	5	3	3	4	5	6	5	5	3									46
1	B					2	2	7	8	8	4	5	8	3													47
1	C				1	5	9	5	6	7	4	8	2														47
1	D				1	6	5	3	7	1	6	6	5	5	1	1											47
2	A	1		1	2		1	3	5	2	4	2	3	2	1	1		2	1	2	3	3		2		1	43
2	B								3		3	4	4	7	4	4	2	2	3	2	3	1	2				41
2	C								1		1	3	2	1	3	5	7	10	4	2	3		1	1			44
2	D									1		3	1	3	2	2	6	8	9		4	1	3		1		44
3	A								4	4	9	8	12	5	0	1	1		1								45
3	B						1	1	2	6	4	6	6	8	5	3	2				1						45
3	C							1	5	5	4	12	3	2	5	2	1	1	1	1							43
4	A					1	2	3	7	6	4	8	5	2	4				1								43
4	B								3	2	8	7	9	7	2	5											43
4	C									5	5	16	7	8	2												43
5	A						2	7	16	9	9	4	1	1													49
5	B			4	5	8	6	9	4	9		2	1					1									50
5	C			2	2	6	2	8	2	7	14	6	1														50
6	A						3	5	11	4	8	7	6	3													47
6	B						2	5	8	6	10	9	2	5													47
6	C								2	5	12	4	10	8	1	3	2										47
7	A				1	4	5	4	5	11	15	5															50
7	B	1	1	2	3	6	5	11	8	4	5	3		2													51
7	C			1	1	5	8	9	10	6	2	7	2														51
8	A					2	1	1	6	2	10	5	12	3	3												45
8	B						1	1	2	3	19	9	8	1													44
8	C						3		2	2	3	10	14	8	3												45
9	A								5	7	4	8	11	7	4	1	1										48
9	B					2		4	15	8	7	10	1	1													49
9	C							3	8	9	9	11	7	2													49
9	D							5	18	5	3	3	6	5	1	2											48
10	A							5	6	8	13	10	5	1													48
10	B					2	3	7	6	2	2	4	7	7	5												45
10	C					1	4	5	7	3	14	10	2														46
10	D						1	6	6	1	7	9	7	4	2	3	1										47
11	A						2	7	20	12	4	1															46
11	B								4	5	6	7	7	11	4	2											46
11	C							3	6	7	5	6	7	10	1	1											46
12	A							3	4	9	14	10	4														44
12	B					2	7	4	2	11	3	3	2	4	4	1	1										44
12	C							1	2	2	16	11	4	7	1	1											45
13	A							1	1	2	4	6	8	5	8	6	2	1	2	3							49
13	B						4	4	5	4	5	4	7	3	2	2	1	3		1	2						49
13	C								3	2	7	7	10	4	5	3	3	1	1	1							47
14	A						1	5	8	8	8	8	6	1	2	1											49
14	B					1	1		1	4	4	5	3	5	2	6	8	4	4								48
14	C		1	1		1	1	4	3	7	7	4	6	1	6	1	2	1	1								47
15	A						2	3	1	4	8	5	10	4	3	4	1										45
15	B						3	4	5	4	8	8	3	5	5	2											47
15	C			7	5	7	6	7	6	3		1	2														44
16	A							1	10	16	11	5	1	1													45
16	B						3	4	8	12	9	7															43
16	C						3	4	14	11	12	1															45
Sum		2	2	16	22	63	101	175	303	286	356	326	254	177	98	68	46	37	28	12	16	5	6	3	1	1	2405

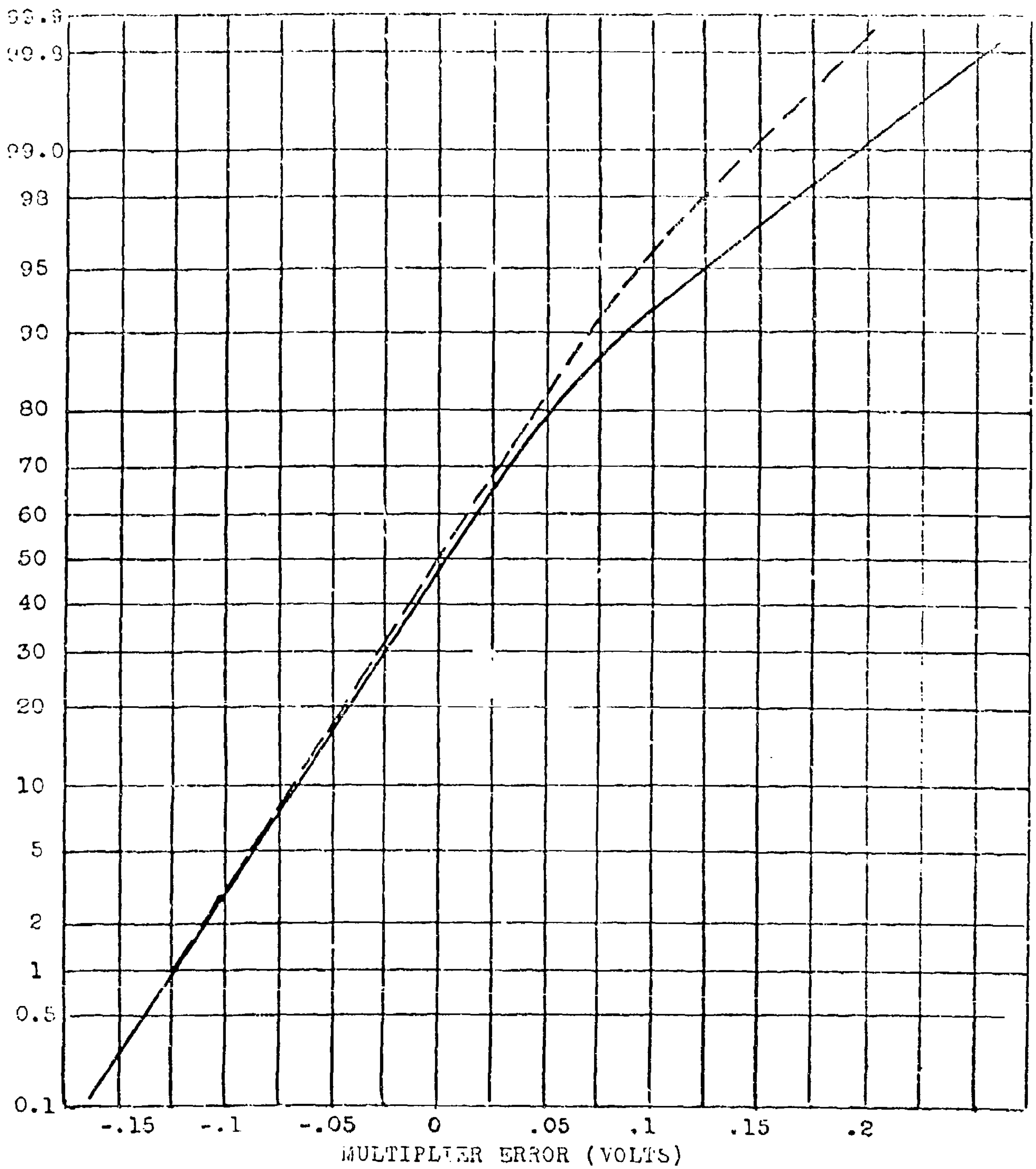


FIGURE 19 CUMULATIVE MULTIPLIER ERROR DISTRIBUTION

The frequencies for all pots of all servos were combined into a single distribution with equal weight. This process could be justified by either of two assumptions: the error voltage is completely uncorrelated with shaft position, or all shaft positions are equally probable. The first of these has to do with the nature of the servo, and the second with the nature of the problem being solved. It would appear that both assumptions hold in the present case. As to the first, the error appears to be uncorrelated with shaft position so long as the padding resistors are correctly adjusted. From the records taken, it appears that they were so adjusted for the servos used. Any remaining correlation would have to be repeatable characteristic of the pot winding machine, and no such characteristic appears on cursory examination of the records. Of course, the 10-cycle periodicity appears on some, but not on others, so it was concluded that there was no important correlation between shaft position and error voltage. Matters of this kind should, of course, be examined by application of statistical techniques to the data, and in fact a start was made in this direction. For the reason cited earlier, however, it wa. decided that the simple procedure described would be adequate.

As regards the second assumption above, that all shaft positions are equally probable, this appears to be true in a general way for the coordinate conversion simulation. For other applications, of course, the situation might be different. In either the quaternion or direction cosine method it is true that, regardless of the orientation, some of the quantities will be large and others small. A complete rotation of the coordinate system causes certain of the multipliers to sweep through their entire range. From such considerations, the hypothesis that all shaft positions are equally likely appears reasonably sound.

All the data of Table 3 were combined into a single cumulative distribution which is shown as the solid curve of Figure 19. This has been plotted on cumulative probability paper so that a normal distribution would appear as a straight line. It may be seen that this curve does not pass through the 50 per cent point at zero error (its mean is not zero) and that it departs from linearity for positive errors. This suggests a bias in the data. An examination of the data of Table 3 shows that Servo 2 has something wrong with it. All the pots have a large positive bias, and, especially Pot A has an unusually large dispersion. If the data for this servo are deleted, the dashed curve of Figure 19 is obtained. It may be seen that now the curve passes through the 50 per cent point at zero error, and it is much more nearly a straight line. There is still some deviation from linearity, and this is no doubt due to bias in some of the pots, but it appears that the errors may be represented reasonably well by a normal distribution. The standard deviation may be determined from the value at which the curve crosses the 84.15 per cent line. This turns out to be 0.053 volts.

In the error analysis, then, it may be assumed that the multiplier errors are normally distributed, with a standard deviation of 0.053 volts, when the voltage across the multiplying pot is ± 100 V. It would be proportionately less for smaller pot voltages.